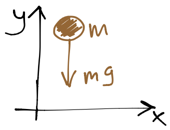
Fritt fall



$my'' = -mg$

$y'' = -g \Rightarrow y' = -gt + C \Rightarrow y = -\frac{gt^2}{2} + Ct + D$

C och D kan bestämmas med hjälp av begynnelsevärdena $y(0)$, $y'(0)$.

Med luftmotstånd

$my'' = -mg - ky'$

$y'' = -g - \frac{k}{m} \cdot y'$

Sätt $y' = v$

multiplicera med $e^{\frac{k}{m} \cdot t}$

$v' + \frac{k}{m} \cdot v = -g$

$e^{\frac{k}{m} t} \cdot v' + e^{\frac{k}{m} t} \cdot \frac{k}{m} \cdot v = -g \cdot e^{\frac{k}{m} \cdot t}$

$(e^{\frac{k}{m} t} \cdot v)' = -g e^{\frac{k}{m} t}$

$e^{\frac{k}{m} t} \cdot v = -g e^{\frac{k}{m} t} \cdot \frac{m}{k} + c$

$(fg)' = f'g + fg'$

Begynnelsevärden $v(0) = 0$

$0 = -\frac{gm}{k} + c$

$c = \frac{gm}{k}$

$v = -\frac{gm}{k} + \frac{gm}{k} e^{-\frac{k}{m} t}$

$\lim_{t \to \infty} v(t) = -\frac{gm}{k}$

$y' = v = -\frac{gm}{k}\left(1 - e^{-\frac{k}{m} t}\right)$

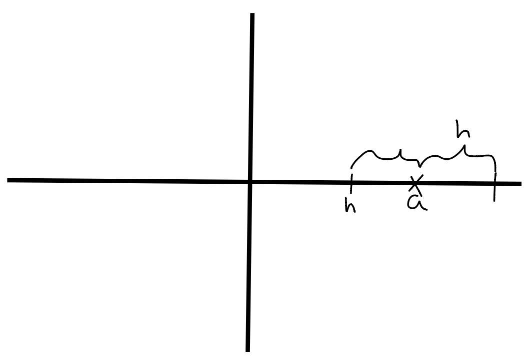
$y = -\frac{gm}{k}\left(t + e^{-\frac{k}{m} t} \cdot \frac{m}{k}\right) + D$ [D bestämms av vilken höjd vi gjorde släppet ifrån.]

DEF absolutbeloppet $|x|$

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

Ex.

$|x - a| \leq h$



$x - a \leq h$ om $x - a \geq 0$

$-(x - a) \leq h$ om $x - a < 0$

$x \leq a + h$

$x - a \geq -h$

$x \geq a - h$

SATS

$|xy| = |x| \cdot |y|$

$|x + y| \leq |x| + |y|$ (triangelolikheten)

OBS!

$|x| = \sqrt{x^2}$

$|x-y|$ är avståndet mellan x och y. $x > y \Rightarrow |x-y| = x-y$

$x < y \Rightarrow |x-y| = y-x$

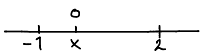
Ex.

Lös ekvationen $|2x+2| = |x-2|$

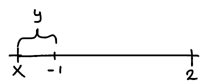
Metod I

$|2x+2| = 2|x+1| = x-2$

avståndet mellan $(x,2) = 2 \cdot \text{avst}(x,-1)$



$\boxed{x=0}$



$2y = y+3$

$y = 3$

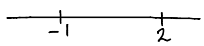
$\boxed{x=-4}$

Metod II

Ta bort beloppstecknen, olika fall.

$$\begin{cases} x \geq -1 \\ x < -1 \end{cases} \Rightarrow x < -1,\ -1 \leq x \leq 2,\ x \geq 2$$

$$\begin{cases} x \geq 2 \\ x < 2 \end{cases}$$



$x < -1$

$-2x-2 = -(x-2)$

$-2x-2 = -x+2 \Rightarrow -x = 4 \Rightarrow x = -4$

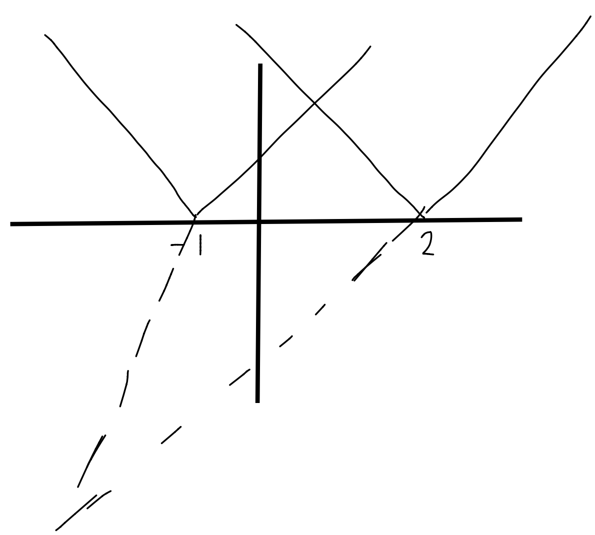
$-1 \leq x < 2$

$2x+2 = -(x-2)$

$2x+2 = -x+2 \Rightarrow 3x = 0 \Rightarrow x = 0$

$x \geq 2$

$2x+2 = x-2 \Rightarrow x = -4$ men $-4 \not\geq 2$

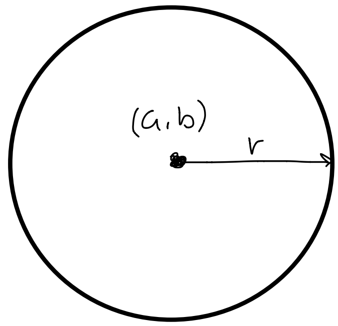


Räta linjer

Kom ihåg $Ax+By+C=0$, (A,B) är normal

Cirkel

Centrum (a,b), radie r



$(x-a)^2+(y-b)^2=r^2$

Ex

$x^2+y^2=x$

$x^2-x+y^2=0$

$\underbrace{\left(x-\frac{1}{2}\right)^2}_{x^2-x+\frac{1}{4}}+y^2=\frac{1}{4}$

cirkel med centrum $\left(\frac{1}{2},0\right)$ och radie $\frac{1}{2}$

Parabel

$y=-2x^2+4x+10$

$=-2(x^2-2x)+10$

$=-2(x-1)^2+2+10$

$=-2(x+1)^2+12$

Trigonometriska identiteter

Kan återföras på Eulers identitet. $e^{ix}=\cos x+i\sin x$

Ex

$e^{i(x+y)}=\cos(x+y)+i\sin(x+y) = e^{ix}\cdot e^{iy}=(\cos x+i\sin y)(\cos y+i\sin y)= \cos x\cdot\cos y+i\sin x\cdot\cos y+i\cos x\cdot\sin y-\sin x\cdot\sin y$

$= \underbrace{(\cos x\cos y-\sin x\sin y)}_{\cos(x+y)}+i\underbrace{(\sin x\cos y+\cos x\cdot\sin y)}_{\sin(x+y)}$

Ex

$e^{3ix}=\cos 3x+i\sin 3x=(e^{ix})^3$

$(e^{ix})^3=(\cos x+i\sin x)^3=\cos^3 x+3i\cos^2 x\cdot\sin x-3\cos x\cdot\sin^2 x-i\sin^3 x$

$=\underbrace{(\cos^3 x-3\cos x\cdot\sin^2 x)}_{\cos^3 x}+i\underbrace{(3\cos^2 x\cdot\sin x-\sin^3 x)}_{\sin^3 x}$

Faktorsatsen

P är ~~ett~~ polynom.

$P(a)=0 \Leftrightarrow P(x)=(x-a)q(x)$ dvs. $(x-a)|P$

Bevis

$\Leftarrow$ $P(a)=0\cdot q(a)=0$

$\Rightarrow$ divisionsalgoritmen ger att $P(x)=(x-a)k(x)+r(x)$ där graden av r < graden av $(x-a)$, dvs r är konstant.

$x=a$ ger $0=0\cdot k(x)+r \Rightarrow r=0$

Ex

$x^3-4x^2+5x-2=0$ Vi ser att $x=1$ är en lösning.

$x-1$ delar VL.

$$
\begin{array}{l|l}
\multicolumn{1}{l}{x^2-3x+2} & \\
\hline
x^3-4x^2+5x-2 & x-1 \\
\underline{-(x^3-x^2)} & \\
-3x^2+5x-2 & \\
\underline{-(-3x^2+3x)} & \\
2x-2 & \\
\underline{-(2x-2)} & \\
0 &
\end{array}
$$

$x^3-4x^2+5x-2=(x-1)(x^2-3x+2)$

$x^2-3x+2=0$

$x=\frac{3}{2}\pm\sqrt{\frac{9}{4}-\frac{8}{4}}$

$x=\frac{3}{2}\pm\sqrt{\frac{1}{4}}$

$x=\frac{3}{2}\pm\frac{1}{2}$

$x_1=2$

$x_2=1$

$x_3=1$

Partialbråksuppdelning

Låt $\frac{P(x)}{q(x)}=\frac{P(x)}{q_1(x)q_2(x)}$ grad $P<$ grad q och $\mathrm{sgd}(q_1,q_2)=1$

Då är $\frac{P(x)}{q(x)}=\frac{P_1}{q_1}+\frac{P_2}{q_2}$ grad $P(k)<$ grad $q(k)$ $k=1.2$

Ex

$\frac{1}{x^2-1}=\frac{1}{(x+1)(x-1)}=\frac{A}{x+1}+\frac{B}{x-1}$

$\frac{A(x-1)+B(x+1)}{(x+1)(x-1)}=\frac{(A+B)x+B-A}{(x+1)(x-1)}$

$\left.\begin{array}{l} A+B=0 \\ B-A=1 \end{array}\right\}$ $\begin{array}{l} B=-A \\ -2A=1 \end{array}$ $\Rightarrow A=-\frac{1}{2}$

$B=\frac{1}{2}$

Partialbråksuppdelning

q_1, q_2 relativt prima.

$\frac{p}{q_1 q_2} = \frac{p_1}{q_1} + \frac{p_2}{q_2}$, alla täljare är av lägre grad än nämnarna.

Bevis

$\frac{p}{q_1 q_2} = [\text{Bezouts id}] = \frac{p(uq_1+vq_2)}{q_1 q_2} = \frac{puq_1}{q_1q_2} + \frac{pvq_2}{q_1q_2} = \frac{pu}{q_2} + \frac{pv}{q_1} = k_2 + \frac{p_2}{q_2} + k_1 + \frac{p_1}{q_1}$, k_2 måste vara lika med $-k_1$ ty V.L $\to 0$ när $x \to \infty$ och $\frac{p_1}{q_1}, \frac{p_2}{q_2}$ likaså.

Ex. $\frac{x^4+2x^3+x^2-2x}{x^3+x^2-x-1} = r$

$$\begin{array}{l}
\quad x + 1 \\
x^4+2x^3+x^2-2x \;\big|\; x^3+x^2-x-1 \\
-(x^4+x^3-x^2-x) \\
\hline
\qquad x^3+2x^2-x \\
\quad -(x^3+x^2-x-1) \\
\hline
\qquad\qquad x^2+1
\end{array}$$

$r(x) = x+1 + \frac{x^2+1}{x^3+x^2-x-1}$

Faktorisera $x^3+x^2-x-1 = q(x)$

$q(1) = 0 \Rightarrow$ det går att dividera

$$\begin{array}{l}
\quad x^2+2x+1 \\
x^3+x^2-x-1 \;\big|\; x-1 \\
-(x^3-x^2) \\
\hline
\quad 2x^2-x \\
\quad -(2x^2-2x) \\
\hline
\qquad x-1 \\
\quad -(x-1) \\
\hline
\qquad 0
\end{array}$$

$q(x) = (x^2+2x+1)(x-1) = (x-1)(x+1)^2$

$r(x) = x+1+\frac{x^2+1}{(x-1)(x+1)^2} = x+1+\frac{A}{x-1}+\frac{bx+c}{(x+1)^2} = x+1+\frac{A}{x-1}+\frac{b(x+1)+(c-b)}{(x+1)^2} = x+1+\frac{A}{x-1}+\frac{b}{x+1}+\frac{c-b}{(x+1)^2} = x+1+\frac{A}{x-1}+\frac{B}{x+1}+\frac{C}{(x+1)^2}$

$= x+1+\frac{A(x+1)^2+B(x+1)(x-1)+C(x-1)}{(x-1)(x+1)(x+1)}$

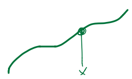
x^2-termer: $A+B = 1$
x-termer: $2A+C = 0$
konst: $A-B-C = 1$

$C = -2A$
$3A-B = 1$
$A+B = 1$
$4A = 2 \quad \Rightarrow \quad A = \frac{1}{2},\ B = \frac{1}{2},\ C = -1 \Rightarrow r(x) = x+1+\frac{1}{2(x-1)}+\frac{1}{2(x+1)}-\frac{1}{(x+1)^2}$

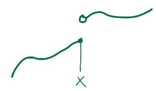
Def

En funktion kallas kontinuerlig om tillräckligt små ändringar i x ger godtyckligt givna små förändringar i $y = f(x)$.

Kontinuerlig

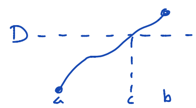


Diskontinuerlig



Satsen om mellanliggande värden

Om $f: \mathbb{R} \to \mathbb{R}$ är kontinuerlig och $f(a)<f(b)$ $(f(a)>f(b))$, $a<b$, och $f(a)< D<f(b)$ så finns ett c
$a<c<b$, $f(c)=D$

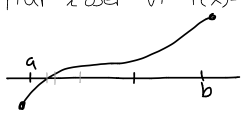


Inte sant om $f: \mathbb{Q} \to \mathbb{Q}$

Ex $f(x)=x^2$
$f(1)=1$
$f(2)=4$
$1<2<4$ men $\nexists x$. så $x^2=2$

Intervallhalvering

Om $f(a)<0$ och $f(b)>0$ finns ett x s.a. $f(x)=0$.
Hur löser vi $f(x)=0$?



Tag medelpunkten $m=\frac{a+b}{2}$, minst ett av intervallen $(a,m]$ och $[m,b)$ har olika tecken i ändpunkterna (om inte $f(m)=0$). Upprepa!

Def

Derivatan $f'(x)$ är $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ om gränsvärdet existerar.

Ex $f(x)=x^2$

$$\frac{f(x+h)-f(x)}{h} = \frac{(x+h)^2-x^2}{h} = \frac{x^2+2hx+h^2-x^2}{h} = 2x+h \to 2x$$

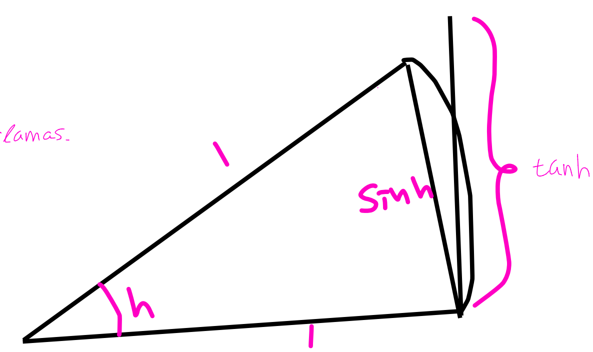
$f(x)=\sin x$, $x=0$

$$\frac{\sin(0+h)+\sin(h)}{h} = \frac{\sin h - \sin 0}{h} = \frac{\sin h}{h} \to 1$$

Bevis

Antag först $h>0$

Jämför sektorns area med trianglarnas.



$$\frac{1 \cdot \sin h}{2} < \frac{h}{2\pi} < \frac{1 \cdot \tan h}{2}$$

$\sin h < h < \tan h = \frac{\sin h}{\cos h}$

$\frac{\sin h}{h} < 1 < \frac{\sin h}{h} \cdot \frac{1}{\cos h}$

$\cos h < \frac{\sin h}{h} < 1$

$h \to 0$, $\sin h \to 1$

Anm.

Gränsvärden när $x \to \infty$

$\log_a x$ växer långsammare än

x^α — " — $\alpha > 0$

x^β om $\beta > \alpha$ — " —

c^x

$\frac{x^2}{e^x} \to 0, \ x \to \infty$

$\frac{\ln x}{x} \to 0, \ x \to \infty$

$\frac{x^2+x}{x^3+1} = \frac{x^2(1+\frac{1}{x})}{x^3(1+\frac{1}{x^3})} \to 0$

$\frac{x^2+x}{x^2+2x} = \frac{x^2(1+\frac{1}{x})}{x^2(1+\frac{2}{x})} \to 1$

Derivatan mäter ändringstakt

1) Hastighet

$S(t)$: tillryggalagd sträcka

$\frac{S(t+h)-S(t)}{h}$, medelhastighet i tidsintervallet $[t, t+h]$

$S'(t)$: hastigheten i t

2) Marginalskatt

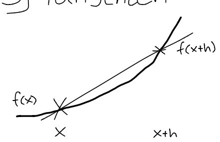
$S(k)$ = Skatt att betala på k kronor

$S(k+h) - S(k)$ = skatt att betala på de sista h kronorna.

$\frac{S(k+h)-S(k)}{h}$ = Skatt/ krona på sista h

$S'(k)$ marginalskatten

3) Tangenten



Kordans lutning : $\frac{f(x+h)-f(x)}{(x+h)-x} = \frac{f(x+h)-f(x)}{h}$, $h \to 0$ $f'(x)$ = tangentens lutning

4) Approximativa beräkningar i samband med mätfel

$\lim_{h \to 0} \frac{f(x+h)-f(x)}{h} = f'(x)$, om h är litet: $\frac{f(x+h)-f(x)}{h} \approx f'(x)$ $\quad f(x+h) \approx f(x) + f'(x)h$

Ex

$(2{,}01)^2$

$f(x) = x^2$

$f'(x) = 2x$

$x = 2$

$(2{,}01)^2 = 2^2 + 2 \cdot 2 \cdot 0{,}01 = 4{,}04$

Exakt

$(2{,}01)^2 = (2+0{,}01)^2 = 4 + 4 \cdot 0{,}01 + 0{,}01^2 = 4{,}0401$

P6)

5. $16x^4-8x^2+1=0$

Sätt $x^2=u$

$16u^2-8u+1=0$

$u^2-\frac{1}{2}u+\frac{1}{16}=0$

$u=\frac{1}{4}\pm\sqrt{\frac{1}{16}-\frac{1}{6}}$

$\left.\begin{array}{l} u_1=\frac{1}{4} \\ u_2=\frac{1}{4}\end{array}\right\}$ $x_{1,2}=\pm\sqrt{\frac{1}{4}}=\frac{1}{2}$ dubbelrötter

Faktorisering av polynomee: $16x^4-8x^2+1=0$

$\Longleftrightarrow$

$16\left(x-\frac{1}{2}\right)^2\left(x+\frac{1}{2}\right)^2$

7) $x^3+1=0$

$x^3=-1$

$x=-1$ är en rot

$$\begin{array}{l} \quad x^2-x+1 \\ \quad x^3+1 \;|\; x+1 \\ -(x^3+x^2) \\ \quad -x^2+1 \\ -(-x^2-x) \\ \quad x+1 \\ -(x+1) \\ \quad 0 \end{array}$$

x^2-x+1

$x=\frac{1}{2}\pm\sqrt{\frac{1}{4}-\frac{4}{4}}$

$x=\frac{1}{2}\pm\sqrt{\frac{-3}{4}}=\frac{1}{2}\pm\sqrt{\frac{i\sqrt{3}}{2}}$

$x_1=-1$

$x_2=\frac{1}{2}+\sqrt{\frac{i\sqrt{3}}{2}}$

$x_3=\frac{1}{2}-\sqrt{\frac{i\sqrt{3}}{2}}$

$x^3+1=(x+1)(x^2-x+1)=(x+1)\left(x-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)\left(x-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)$

13) $\frac{x^3-1}{x^2-2}$

$$\begin{array}{l} \quad x \\ \quad x^3-1 \;|\; x^2-2 \\ -(x^3-2x) \\ \quad 2x-1 \end{array}$$

$x+\frac{2x-1}{x^2-2}=x+\frac{2x-1}{(x-\sqrt{2})(x+\sqrt{2})}=x+\frac{A}{x-\sqrt{2}}+\frac{B}{x+\sqrt{2}}=x+\frac{A(x+\sqrt{2})+B(x-\sqrt{2})}{(x-\sqrt{2})(x+\sqrt{2})}=\frac{(A+B)x+(A-B)\sqrt{2}}{(x+\sqrt{2})(x-\sqrt{2})}$

$A+B=2$ $\quad B=2-A$ $\quad A=1+\frac{1}{2\sqrt{2}}$

$(A-B)\sqrt{2}=-1$ $\quad 2A-2=\frac{1}{\sqrt{2}}$ $\quad B=2-\left(1+\frac{1}{2\sqrt{2}}\right)=1-\frac{1}{2\sqrt{2}}$

$\frac{x^3-1}{x^2-2}=x+\left(1+\frac{1}{2\sqrt{2}}\right)\frac{1}{x-\sqrt{2}}+\left(1-\frac{1}{\sqrt{2}}\right)\frac{1}{x+\sqrt{2}}$

19) $x^2+px+q=0$

$x=\frac{-p}{2}\pm\sqrt{\underbrace{\left(\frac{p}{2}\right)^2-q}_{D}}$ vad händer om $D<0$..? $\Longrightarrow$ $x=\frac{-p}{2}\pm i\sqrt{|D|}$

Om $z=u+iv$, definierar vi konjugatet $\bar{z}=u-iv$

SATS
Om p är ett polynom med reella koefficienter och $p(z)=0$ är även $p(\bar{z})=0$

SATS
$z=x+iy$, $w=u+iv$

$\overline{z+w}=\bar{z}+\bar{w}$

$\overline{z\cdot w}=\bar{z}\cdot\bar{w}$

Bevis
$\overline{z+w}=\overline{x+iy+u+iv}=\overline{(x+u)+i(y+v)}=(x+u)-i(y+v)$

$\bar{z}+\bar{w}=x-iy+u-iv=(x+u)-i(y+v)$

$\overline{z\cdot w}=\overline{(x+iy)(u+iv)}=\overline{xu+iyu+ixv-yv}=xu-yv-i(yu+xv)$

$\bar{z}\cdot\bar{w}=(x-iy)(u-iv)=xu-iyu-ixv-yv=xu-yv-i(yu+xv)$

Bevisa att $P(\bar{z})=0$

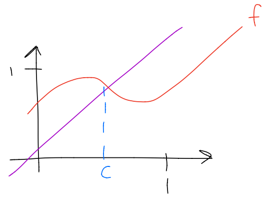
$P(z)=a_nz^n+a_{n-1}z^{n-1}+\dots+a_2z^2+a_1z+a_0 \qquad a_k\in\mathbb{R}$

$\overline{P(z)}=\overline{a_nz^n+\dots+a_0}=\overline{a_nz^n}+\overline{a_{n-1}z^{n-1}}+\dots+\overline{a_2z^2}+\overline{a_1z}+\overline{a_0}=\bar{a}_n\overline{z^n}+\bar{a}_{n-1}(\overline{z^{n-1}})+\dots+\bar{a}_1\bar{z}+\bar{a}_0=a_n(\bar{z}^n)+a_{n-1}(\bar{z}^{n-1})+\dots+a_1\bar{z}+a_0=P(\bar{z})$

Om nu $P(z_0)=0$ så är $p(\bar{z}_0)=\bar{0}=0$

1.4 32)

$f:[0,1]\to[0,1]$ och är kontinuerlig. Då finns $c\in[0,1]$ s.a. $f(c)=c$



Bevis
Betrakta $g(x)=f(x)-x$

$g(0)=f(0)-0\geq 0$

$g(1)=f(1)-1\leq 0$

Antingen: $g(0)=0$ (fix punkt)

$g(0)>0$

då gäller antingen: $g(1)=0$ (fixpunkt)

$f(1)=1$

eller: $g(0)>0$

$g(1)<0$

Enligt satsen om mellanliggande värden

$\exists c$ s.a. $g(c)=0$

$f(c)=c$ (fix punkt)

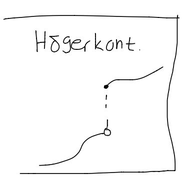
In mathematics, a **fixed point** (sometimes shortened to **fixpoint**, also known as an **invariant point**) of a function is an element of the function's domain that is mapped to itself by the function. A set of fixed points is sometimes called a *fixed set*. That is to say, *c* is a fixed point of the function *f*(*x*) if and only if *f*(*c*) = *c*. This means *f*(*f*(...*f*(*c*)...)) = $f^n(c)$ = *c*, an important terminating consideration when recursively computing *f*. For example, if *f* is defined on the real numbers by

$$f(x)=x^2-3x+4,$$

then 2 is a fixed point of *f*, because *f*(2) = 2.

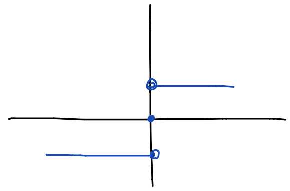
34)
f udda dvs. $f(-x)=-f(x)$ $\forall x$ Speciellt $x=0$ $f(0)=-f(0)$, $f(0)=0$

Om f är högerkontinuerlig, dvs $\lim\limits_{\substack{x\to a\\ x>0}} f(x)=f(a)$ $\left(\lim\limits_{x\to a^+} f(x),\ \lim\limits_{x\downarrow a} f(x)\right)$ $a=0$, så är f i själva verket kontinuerlig.



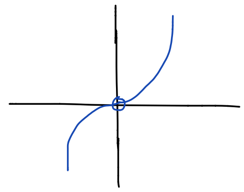
Ty: Låt $x<0$ $\lim\limits_{x\to 0^-} f(x) \underset{\text{Ty udda}}{=} \lim\limits_{x\to 0} -f(-x) = \lim\limits_{y\to 0^+} -f(y) \underset{\text{Ty högerkont}}{=} -f(0)=0$

Def $\operatorname{sgn} x \begin{cases} 1, & x>0 \\ -1, & x<0 \\ (0, & x=0) \end{cases}$



$x = |x| \operatorname{sgn} x$

$g(x) = x^2 \operatorname{sgn} x \begin{cases} x^2, & x>0 \\ -x^2, & x<0 \\ ?, & x=0 \end{cases}$



g blir kont om
$g(0)=0$ dvs. $\operatorname{sgn} 0$ spelar ingen roll

Är g deriverbar i 0?
Vi gissar att $g'(0)=0$

Bevis
$\left|\frac{g(x)-g(0)}{x}\right| = \left|\frac{g(x)}{x}\right| = \frac{x^2}{|x|} = |x| \to 0$ när $x \to 0$

Räkneregler

1. Derivering är en linjär operation. Dvs: om a, b konst $\Rightarrow (af+bg)' = af' + bg'$
2. Produktregeln: $(fg)' = f'g + fg'$
3. Kedjeregeln: $\frac{d}{dx} f(g(x)) = f'(g(x))g'(x)$

Ex

$$\frac{d}{dx}(x^2 + x\sin x) = 2x + 1\sin x + x\overset{(\sin x)'}{\cos x}$$

$$\frac{d}{dx} x e^{\sin x^2} = 1 \cdot e^{\sin x^2} + x e^{\sin x^2} \cdot (\cos x^2) \cdot 2x = e^{\sin x^2} \cdot (1 + 2x^2 \cdot \cos x^2)$$

Partiella derivator

$\frac{\partial f}{\partial x}$ = derivera map x men behåll övriga variabler konstanta.

Ex

$$\frac{\partial}{\partial x} x e^{x^2 y} = e^{x^2 y} + x e^{x^2 y}(2xy)$$

$$\frac{\partial}{\partial y} x e^{x^2 y} = x e^{x^2 y}(x^2).$$

Bevis

1. Lätt
2. $\frac{f(x+h)g(x+h) - f(x)g(x)}{h} = \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h} = g(x+h)\overbrace{\frac{f(x+h)-f(x)}{h}}^{f'(x)} + f(x)\overbrace{\frac{g(x+h)-g(x)}{h}}^{g'(x)} = g(x)f'(x) + f(x)g'(x)$
3. $\frac{f(g(x+h)) - f(g(x))}{h} = \frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)} \cdot \frac{g(x+h) - g(x)}{h} = f'(g(x)) \cdot g'(x)$

Grundläggande derivator

$$\frac{d}{dx}\sin x = \lim_{h\to 0} \frac{\sin(x+h) - \sin(x)}{h} = \frac{\sin(x)\cos h + \cos x \sin h - \sin x}{h} = \sin(x) \cdot \frac{\cos h - 1}{h} + \cos x \cdot \frac{\sin h}{h} = \sin(x) \cdot \frac{(\cos h - 1)(\cos h + 1)}{h(\cos h + 1)} + \cos(x) \cdot \frac{\sin h}{h} = \sin x \cdot \frac{\cos^2 h - 1}{h(\cos h + 1)} + \cos(x) \cdot \frac{\sin h}{h} =$$

$$-\sin x \cdot \frac{\sin^2 h}{h(\cos^2 h + 1)} + \cos x \cdot \frac{\sin h}{h} \to \cos(x)$$

$$-\sin x \cdot \underbrace{\frac{\sin^2 h}{h}}_{0} \cdot \frac{1}{\underbrace{(\cos^2 h + 1)}_{2}} + \cos x \cdot \underbrace{\frac{\sin h}{h}}_{1}$$

$$\frac{d}{dx}\cos(x) = \frac{d}{dx}\left(\sin \frac{\pi}{2} - x\right) = \cos\left(\frac{\pi}{2} - x\right)(-1) = -\sin(x)$$

$$\frac{d}{dx}\tan(x) = \overset{\text{Kvotregeln}}{\left(\frac{f}{g}\right)' = (f(g^{-1}))' = f'g^{-1} + f(-1)g^{-2}g' = \frac{f'}{g} - \frac{fg'}{g^2} = \frac{f'g - fg'}{g^2}} = \frac{d}{dx}\frac{\sin x}{\cos x} = \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2(x)} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

Kedjeregeln ger för inversa funktionerna:

$$f^{-1}(f(x)) = x$$

$$f^{-1\prime}(f(x))f'(x) = 1 \Rightarrow (f^{-1})' = \frac{1}{f'}$$

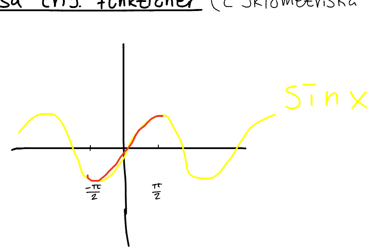
$$\frac{d}{dx} a^x = \lim_{h\to 0} \frac{a^{x+h} - a^x}{h} = \lim_{h\to 0} \frac{a^x a^h - a^x}{h} = a^x \lim_{h\to 0} \frac{a^h - 1}{h}$$ om vi väljer $a = e$ blir $\lim_{h\to 0} \frac{a^h - 1}{h} = 1$ (bevisas ej)

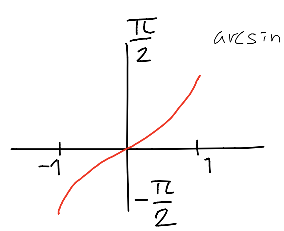
$$\frac{d}{dx} e^x = e^x$$

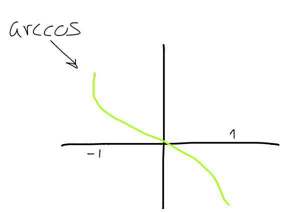
Inversen $\ln y$, $y = e^x$

$$\ln'(y) = \ln'(e^x) = \frac{1}{e^x} = \frac{1}{y} \qquad \frac{d}{dx}\ln(x) = \frac{1}{x}$$

"Inversa" trig. funktioner (Cyklometriska funktioner)







Derivator

$y = \arctan(\tan y) \quad (-\frac{\pi}{2} < y < \frac{\pi}{2})$

$\frac{d}{dx}$: $1 = \arctan'(\tan y)\tan' y = \arctan'(\tan y) \cdot \frac{1}{\cos^2 y}$

$\cos^2 y = \arctan'(\tan y)$

$\sin^2 y + \cos^2 y = 1$

$\frac{\sin^2 y}{\cos^2 y} + \frac{\cos^2 y}{\cos^2 y} = \frac{1}{\cos^2 y}$

$\tan^2 y + 1 = \frac{1}{\cos^2 y}$

$\arctan'(\tan y) = \cos^2 y = \frac{1}{\tan^2 y + 1} \qquad \tan y = x \Rightarrow \arctan'(x) = \frac{1}{1+x^2}$

$y = \arcsin(\sin y)$

$\frac{d}{dx}$: $1 = \arcsin'(\sin y)\cos y$

$\arcsin'(\sin y) = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}} \Rightarrow \arcsin'(x) = \frac{1}{\sqrt{1-x^2}}$

Implicit derivering

Vad är tangenten till cirkeln $x^2 + y^2 = 4$ i punkten $(\sqrt{3}, 1)$?

Tänk att $y = y(x)$. Derivera map x.

$2x + 2yy' = 0$

$y' = -\frac{2x}{2y} = \frac{-x}{y} = \sqrt{3} \Rightarrow$ Tangentens ekvation: $\sqrt{3}(x+\sqrt{3}) = (y-1)$

Explicit

$y^2 = 4 - x^2$

$y = \sqrt{4-x^2} = (4-x^2)^{\frac{1}{2}}$

$y' = \frac{1}{2}\cdot(4-x^2)^{-\frac{1}{2}}(-2x) = \frac{1}{2}\frac{1}{\sqrt{1}}(-2)(-\sqrt{3}) = \sqrt{3}$

Linjär approximation

$f(x+h) - f(x) \approx f'(x)h$

$f(x+h) \approx f(x) + f'(x)h$

Ex $\sin(47°)$

$= \sin(\frac{\pi}{4} + \frac{2\pi}{180}) \approx \sin\frac{\pi}{4} + \cos\frac{\pi}{4}\cdot\frac{2\pi}{180} = \frac{1}{\sqrt{2}}(1 + \frac{\pi}{90})$

Ex: Imp der

Bestäm $\sin y$ där y är lösning till ekvationen $e^y = x + y$, $x = 2 \pm 0.001$

$\sin y = \sin(y(x))$

$\frac{d}{dx}\sin(y(x)) = \cos(y(x))\, y'(x)$

derivera $e^y = x+y$ map x: $e^y y' = 1 + y' \Leftrightarrow e^y y' - y' = 1 \Leftrightarrow y' = \frac{1}{e^y - 1}$

För $x = 2$ ger matlab: $y = 1,1462$

$y' = 0,4659$

$\cos y = 0,4120$

$\sin y = 0,9112$

För $x = 2 \pm 0,01$ får vi: $\sin y = 0,9112 \pm 0,01 \cdot 0,4569 \cdot 0,01 = 0,9112 \pm 0,001919$

I teoretiska sammanhang ersätter man ofta lin.approximationen med medelvärdessatsen.

Medelvärdessatsen

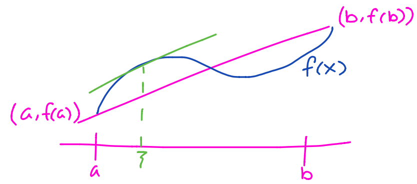
Antag att $f(x)$ är definierbar på (a,b) och kontinuerlig på $[a,b]$. Då finns ξ, $a<\xi<b$, s.a. $f(b)-f(a)=f'(\xi)(b-a)$

Jämförelsevis med lin.app: $f(b)\approx f(a)+f'(a)(b-a)$

Bevis

$\frac{f(b)-f(a)}{b-a}=f'(\xi)$

Lutning för linjen mellan ändpunkterna



Användning

Om $f'(x)=0$ i $[a,b]$ är f konstant.

Bevis

Tag $x\in[a,b]$ $\quad f(x)-f(a)=f'(\xi)(x-a)=0$

$f(x)=f(a)\quad \forall x\in[a,b]$

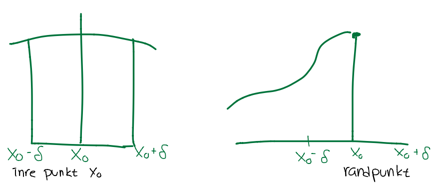
Om $f'(x)>0$ i $[a,b]$ så är $f(b)>f(a)$

Bevis

$f(b)-f(a)=f'(\xi)(b-a)>0$

Def

En funktion f i $D_f\to\mathbb{R}$ sägs ha *lokalt maximum* i punkten x_0 om $f(x_0)\geq f(x)$ för alla x i en omgivning $\{x: |x-x_0|<\delta\}\cap D_f$ för något δ



SATS

Om $f'(x_0)$ existerar i en inre maxpunkt/minpunkt så är $f'(x_0)=0$

Ex

Bestäm max & min av $f(x)=x^3-3x+3$ över $[3,\frac{3}{2}]$

1. Inre punkter, stationära punkter $(f'(x)=0)$

$f'(x)=3x^2-3$

$3x^2-3=0 \Leftrightarrow x^2-1=0 \Leftrightarrow x=\pm1$

2. Randpunkter

$-3,\ \frac{3}{2}$

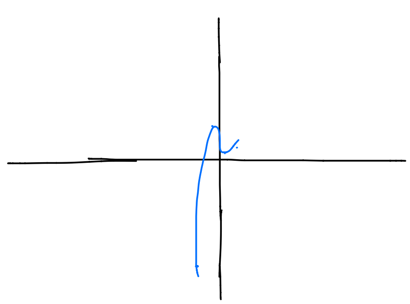
3. Jämför funktionsvärden

$f(-1)=5 \quad\leftarrow$ MAX

$f(1)=1$

$f(-3)=-15 \quad\leftarrow$ MIN

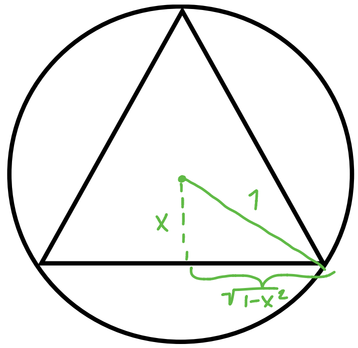
$f(\frac{3}{2})=\frac{15}{8}<2$



Ex

Hur stor är den största likbenta triangeln vilken kan skrivas in i en cirkel med radien 1?

Triangelns bas är $2\sqrt{1-x^2}$
Triangelns höjd är $1+x$
Triangelns area: $A(x)=\frac{2\sqrt{1-x^2}(1+x)}{2}$
$=(1+x)\sqrt{1-x^2}$

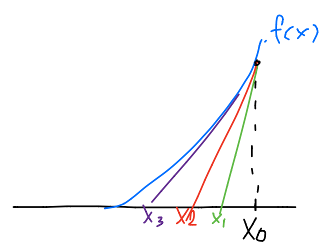


$A'(x)=1(\sqrt{1-x^2})+(1+x)\left(\frac{1}{2}(1-x^2)^{-\frac{1}{2}}(-2x)\right)$
$=\sqrt{1-x^2}-\frac{x(1+x)}{\sqrt{1-x^2}}=\frac{1-x^2-x(1+x)}{\sqrt{1-x^2}}$

$A'(x)=0 \Leftrightarrow 1-x^2-x(1+x)=0$

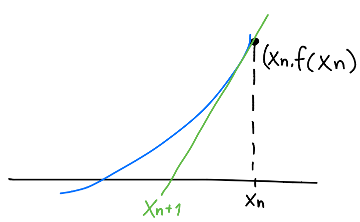
$1-x^2-x-x^2=0$
$2x^2+x-1=0$
$x^2+\frac{x}{2}-\frac{1}{2}=0$
$x=\frac{-1}{4}\pm\sqrt{\frac{1}{16}+\frac{8}{16}}$
$x=\frac{-1}{4}\pm\sqrt{\frac{9}{16}}$
$x=\frac{-1}{4}\pm\frac{3}{4} \Rightarrow x_1=\frac{-1}{4}+\frac{3}{4}=\frac{1}{2}$ $\left(x_2=\frac{-1}{4}-\frac{3}{4}=(-1)\right)$

Detta måste vara ett maximum ty extremerna ger mindre triangel och $x=1$ ger arean $=0$
$A(0)=1$
$A(\frac{1}{2})=\frac{3}{2}\sqrt{1-\frac{1}{4}}=\frac{3}{4}\sqrt{3}>1$



Newtons metod för att lösa $f(x)=0$

Succesivt förbättrade approximationer. x_0 som start



Iterationsformeln

Ekvationen för tangenten i $(x_n, f(x_n))$
$y-f(x_n)=f'(x_n)(x-x_n)$
den skär x-axeln i punkten $(x_{n+1}, 0)$

$-f(x_n)=f'(x_n)(x_{n+1}-x_n)$
$\frac{-f(x_n)}{f'(x_n)}=x_{n+1}-x_n$
$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}$

Logaritmisk derivering

$\frac{d}{dx}\ln f(x)=\frac{1}{f(x)}\cdot f'(x)=\frac{f'(x)}{f(x)}$ (Logaritmiska derivatan av f)

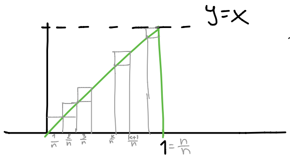
Användning

Låt x-värdet ändras med en faktor $d=(\frac{p}{100})$. Liniär approximation ger då $f(x+dx)\approx f(x)+dxf'(x)$, ändringen i f är $(\approx dx\cdot f'(x),)$ relativa ändringen i f i dx $\frac{f'(x)}{f(x)}$.

P% ändring i f:s värde
q% ändring i g:s värde
ger p+q i fg.

$\frac{(fg)'}{fg}=\frac{fg'+f'g}{fg}=\frac{g'}{g}+\frac{f'}{f}$ log. der(fg) = logder f + logder g

<u>Ett fånigt sätt att räkna ut arean av en triangel.</u>



Triangelns area = A

A kan approximeras med över- & undersummor.

$$\sum_{k=0}^{n-1}\frac{k}{n}\cdot\frac{1}{n} \leqslant A \leqslant \sum_{k=1}^{n}\underbrace{\frac{k}{n}\cdot\frac{1}{n}}_{\text{area av rektangel}} \iff \frac{1}{n^2}\sum_{k=1}^{n-1}k \leqslant A \leqslant \frac{1}{n^2}\sum_{k=1}^{n}k \iff \frac{(n-1)n}{2n^2} \leqslant A \leqslant \frac{n(n+1)}{2n^2} \iff$$

$$\frac{n^2-n}{2n^2} \leqslant A \leqslant \frac{n^2+n}{2n^2} \iff \frac{1}{2}-\frac{1}{2n} \leqslant A \leqslant \frac{1}{2}+\frac{1}{2n} \qquad n\to\infty \Rightarrow \frac{1}{2} \leqslant A \leqslant \frac{1}{2}$$

2.3

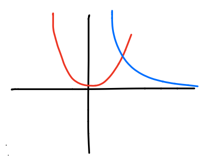
48. Kurvorna $y=x^2$ och $y=\frac{1}{\sqrt{x}}$ skär varandra i rät vinkel.

Kurvornas skärning: $y=x^2=\frac{1}{\sqrt{x}}$

$x^4=\frac{1}{x}$

$x^5=1$

$x=1$ (+ komplexa)



Kurvornas tangenter har koefficienter:

$y'=2x=2$ i $x=1$

$y'=-\left(\frac{1}{2x}\right)^{-\frac{3}{2}}=-\frac{1}{2}$ i $x=1$

Alt 1

Två linjer med r.k. k_1, k_2 är $\perp$ om $k_1 \cdot k_2=-1$. $2\cdot -\frac{1}{2}=-1$

Alt 2

Riktningsvektorerna $(1,k)$ är $\perp$ ty $(1,2)\cdot(1,-\frac{1}{2})=1^2+2\left(\frac{-1}{2}\right)=0$

2.7

11. Ett klots radie ökar med 2%. Hur mycket ökar volymen?

$V=\frac{4\pi}{3}\cdot r^3$

$V'r=4\pi r^2$

Lin. app.

$V\left(r+\frac{2}{100}r\right)\approx V(r)+V'(r)\left(\frac{2}{100}r\right)$

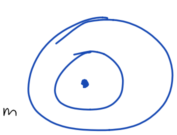
ökningen $\approx V'(r)\frac{2}{100}r$

relativa ändringen: $\frac{1}{V(r)}\cdot V'(r)\cdot\frac{2}{100}\cdot r=\frac{4\pi\cdot r^2\cdot 3\cdot 2\cdot r}{4\pi\cdot r^3\cdot 100}=\frac{6}{100}$

Svar: 6%

4.1

3. En sten släpps i vatten och en cirkulär våg utbreder sig. Hur fort växer arean innanför cirkeln när radien är 20cm och växer med $4\frac{m}{s}$?



Arean $A(r)=\pi r^2$

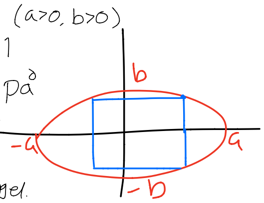
$\frac{dA}{dt}=\frac{dA}{dr}\cdot\frac{dr}{dt}=2\pi r\cdot r'$

$2\pi r\cdot r'=2\pi\cdot 20\cdot 4=160\pi \ \frac{cm^2}{s}$

4.8

13. Ellips $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ $(a>0, b>0)$

Punkten (x,y) på ellipsen beskriver en axelparallell inskriven rektangel.



Dess area (rektangelns) är $2x\cdot 2y=4x\cdot y$

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \Leftrightarrow \frac{y^2}{b^2}=1-\frac{x^2}{a^2} \Leftrightarrow y^2=b^2\left(1-\frac{x^2}{a^2}\right) \Leftrightarrow y=\sqrt{b^2\left(1-\frac{x^2}{a^2}\right)}$, antag $x,y>0$

$A(x)=4xb\sqrt{1-\frac{x^2}{a^2}}=4x\frac{b}{a}\sqrt{a^2-x^2}$

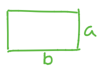
$A'(x) = \frac{4b}{a}\left(\sqrt{a^2-x^2} + x\frac{1}{2}(a^2-x^2)^{-\frac{1}{2}}\cdot(-2x)\right) = \frac{4b}{a}\left(\sqrt{a^2-x^2} - \frac{x^2}{\sqrt{a^2-x^2}}\right) = \frac{4b}{a}\cdot\frac{a^2-2x^2}{\sqrt{a^2-x^2}}$

$A'(x) = 0$ för $a^2-2x^2=0$
$a^2 = 2x^2$
$x^2 = \frac{a^2}{2}$
$x = \underset{(-)}{+} \frac{a}{\sqrt{2}}$

Anmärkning
Ellipsens area är πab

Detta är maximum ty extremerna har arean 0.
$A\left(\frac{a}{\sqrt2}\right) = 4\cdot\frac{a}{\sqrt2}\cdot\frac{b}{a}\left(\sqrt{\underbrace{a^2-\frac{a^2}{2}}_{\frac{a^2}{2}}}\right) = 4\cdot\frac{b}{\sqrt2}\frac{a}{\sqrt2} = 2ab$

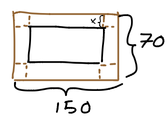
8. Finn den största rektangeln med given omkrets



$A = ab$
$2P = 2a+2b$

$a+b = P$
$b = P-a$
$A(a) = a(P-a) = aP-a^2$
$A'(a) = P-2a$
$A'(a) = 0 \Leftrightarrow P = 2a \Leftrightarrow a = \frac{P}{2}$, $b = P-\frac{P}{2} = \frac{P}{2} = a$, rektangeln är en kvadrat.

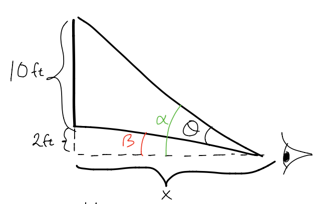
18. Vi gör en låda av ett rektangulärt stycke wellpapp.



Maximera volymen.

Lådans volym
$V(x) = (150-2x)(70-2x)x = 10500x - 440x^2 + 4x^3$
$V'(x) = 10500 - 880x + 12x^2$
$V'(x) = 0 \Leftrightarrow x^2 - \frac{880x}{12} + \frac{10500}{12} = 0 \Leftrightarrow x^2 - \frac{220}{3}x + 875 = 0 \Leftrightarrow x = \frac{110}{3} \pm \sqrt{\left(\frac{110}{3}\right)^2 - 875} = \frac{110}{3} \pm \frac{65}{3}$ $\quad x = \begin{cases} 58+\frac{1}{3} \\ 15 \end{cases}$, $2\times 58+\frac{1}{3}$ kan inte dras ifrån 70. $\Rightarrow x = 15$.
Extremerna $x=0$ och $x=35$ ger båda volym $= 0$.
$V(15) = 72000\ cm^3$

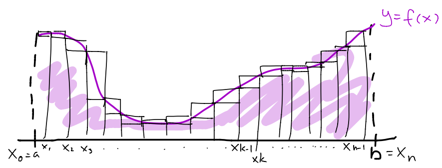
46.



Maximera θ

$\theta = \arctan\frac{12}{x} - \arctan\frac{2}{x}$
$\theta' = \frac{1}{1+\left(\frac{12}{x}\right)^2}\left(-\frac{12}{x^2}\right) - \frac{1}{1+\left(\frac{2}{x}\right)^2}\left(-\frac{2}{x^2}\right) = \frac{-12}{x^2+144} + \frac{2}{x^2+4} = \frac{-12(x^2+4)+2(x^2+144)}{(x^2+4)(x^2+144)} = \frac{240-10x^2}{(x^2+144)(x^2+4)}$
$\theta' = 0$ för $240-10x^2=0$
$240 = 10x^2$
$x^2 = 24$
$x = \underset{(-)}{+} 2\sqrt{6} \approx 4{,}9$
Optimalt $\theta = 45.6^\circ$

Integraler



$a=x_0<x_1<x_2<\ldots<x_{k-1}<x_k<\ldots<x_{n-1}<x_n=b$

Låt m_k och M_k vara tal s.a. $m_k \leqslant f(x) \leqslant M_k$ för $x_{k-1} \leqslant x \leqslant x_k$

Definiera översumman

$U(f,n)=\sum_{k=1}^{n} M_k(x_k-x_{k-1})$

Undersumman

$L(f,n)=\sum_{k=1}^{n} m_k(x_k-x_{k-1})$

Def

Om det finns precis ett tal I som är $\geqslant$ alla undersummor och $\leqslant$ alla översummor så kallas f intergrabel (integrerbar) och $I=\int_a^b f(x)\,dx$.

Sats

Om f är begränsad och styckvis kontinuerlig är f integrerbar. f kan även approximeras godtyckligt väl med en så kallad Riemannsumma. $\sum_{k=1}^{n} f(c_k)(x_k-x_{k-1})$, $x_{k-1} \leqslant c_k \leqslant x_k$

Ex på icke integrabel funktion

Dirichletfunktionen $d(x): [0,1] \to \mathbb{R}$ $\quad d(x)=\begin{cases}1, & x\in\mathbb{Q}\\ 0, & x\notin\mathbb{Q}\end{cases}$

Varje $U \geqslant 1$

Varje $L \leqslant 0$

Sats

1) $\int_a^b cf(x)+dg(x) = c\int_a^b f(x) + d\int_a^b g(x)$

2) $\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$

3) $\left|\int_a^b f(x)\,dx\right| \leqslant \int_a^b |f(x)|\,dx$, triangelolikheten

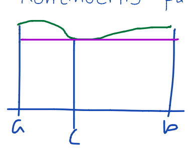
Sats Integralkalkylens huvudsats.

Låt f vara en kontinuerlig funktion och $F(x)=\int_a^x f(t)\,dt$, då är $F'=f$

För beviset behöver (?) vi:

Integralkalkylens medelvärdessats.

f kontinuerlig på $[a,b]$, då finns ett $c\in[a,b]$ s.a. $\int_a^b f(x)\,dx = f(c)(b-a)$



Bevis

Låt $M=\max f(x)$, $x\in[a,b]$

$m=\min f(x)$, $x\in[a,b]$

$m \leqslant f(x) \leqslant M$

$\int_a^b m\,dx \leqslant \int_a^b f(x)\,dx \leqslant \int_a^b M\,dx \iff m(b-a) \leqslant \int_a^b f(x)\,dx \leqslant M(b-a) \iff m \leqslant \frac{\int_a^b f(x)\,dx}{b-a} \leqslant M$

Satsen om mellanliggande värden ger $\exists c$ s.a. $f(c)=\frac{\int_a^b f(x)\,dx}{b-a}$

Bevis av huvudsatsen

$$\frac{F(x+h)-F(x)}{h} = \frac{1}{h}\left(\int_a^{x+h} f(t)dt - \int_a^{x} f(t)dt\right) = \frac{1}{h}\int_x^{x+h} f(t)dt = [\text{m.v-Satsen}] = \frac{1}{h} f(c)(x+h-x) = f(c) \underset{\text{när } h \to 0}{\to} f(x) \qquad x \leqslant c \leqslant x+h$$

Övning: Fullborda beviset.

Följdsats

Låt G vara en primitiv funktion till f, dvs. $G' = f$

Då är $\int_a^b f(t)dt = G(b) - G(a)$

Bevis

$\frac{d}{dx}(F-G) = f - f = 0$

$F - G = C$ (konstant)

$G = F - C$,

$G(b) - G(a) = F(b) - C - (F(a) - C) = F(b) - F(a) = F(b) - 0 = \int_a^b f(t)\,dt$

Ex

$$\int_1^2 x + e^x\,dx = \frac{x^2}{2} + e^x \Big|_1^2 = \frac{2^2}{2} + e^2 - \left(\frac{1^2}{2} + e^1\right) = \frac{4}{2} + e^2 - \frac{1}{2} - e = \frac{3}{2} + e^2 - e$$

$$\frac{d}{dx}\int_0^x \sin^2 t\,dt = \sin x^2$$

$$\frac{d}{dx}\int_x^{x^2} e^{t^2}dt = \frac{d}{dx}\big(F(x^2) - F(x)\big) = F'(x^2)\cdot\underbrace{2x}_{\text{inre derivata}} - F'(x) = e^{x^4}\cdot 2x - e^{x^2}$$

Primitiva funktioner

$\int \frac{1}{1+x^2}\,dx = \arctan x + C$

$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C$

$\int \frac{1}{x}\,dx = \ln|x| + C$, ty $\ln|x| = \begin{cases} \ln x, & x>0 \\ \ln(-x) & x<0 \end{cases}$ $\qquad \frac{d}{dx}\ln|x| = \begin{cases} \frac{1}{x}, & x>0 \\ \frac{1}{-x}(-1) = \frac{1}{x} & x<0 \end{cases}$

Partiell integration

$(fg)' = f'g + fg' \Leftrightarrow f'g = (fg)' - fg' \Leftrightarrow \int f'g\,dx = fg - \int fg'\,dx$

Byt beteckningar

$\int fg\,dx = Fg - \int Fg'\,dx$

Ex

$\int \underset{g}{x}\cdot \underset{f}{\sin x}\,dx = (-\cos x)x - \int(-\cos x)\cdot 1\,dx = -x\cdot\cos x + \sin x + C$

$\int \underset{g}{x^2}\underset{f}{e^x}\,dx = e^x\cdot x^2 - \int e^x 2x\,dx = e^x x^2 - \left(e^x 2x - \int e^x\cdot 2\,dx\right) = e^x x^2 - 2xe^x + 2e^x + C$

Ex. Partiell integration

$$\int x^3 \ln x\,dx = \frac{x^4}{4}\ln x - \int \frac{x^4}{4}\cdot\frac{1}{x}\,dx = \frac{x^4}{4}\ln x - \int \frac{x^3}{4}\,dx = \frac{x^4}{4}\ln x - \frac{x^4}{16} + C$$

$$\int \ln x\,dx = \int 1\cdot \ln x\,dx = x\cdot\ln x - \int x\cdot\frac{1}{x}\,dx = x\cdot\ln x - \int 1\,dx = x\cdot\ln x - x + C$$

för att få $2x$ i täljaren

$$\int \arctan x\,dx = \int 1\cdot\arctan x\,dx = x\cdot\arctan x - \int x\cdot\frac{1}{1+x^2}\,dx = x\cdot\arctan x - \frac{1}{2}\int\frac{2x}{1+x^2}\,dx = x\cdot\arctan x - \frac{1}{2}\ln(1+x^2) + C$$

$$\int \tan x\,dx = \int \sin x\cdot\frac{1}{\cos x}\,dx = -\cos x\cdot\underbrace{\frac{1}{\cos x}}_{\cos x^{-1}} - \int -\cos x\cdot\frac{-1}{\cos^2 x}\cdot -\sin x = -1 + \int\frac{\sin x}{\cos x}\,dx = -1 + \int \tan x\,dx$$ Dåligt

$$\int \tan x\,dx = -\ln|\cos x| \quad \text{ty, } \frac{d}{dx}(-\ln|\cos x|) = \frac{-1}{\cos x}\cdot -\sin x = \frac{\sin x}{\cos x} = \tan x$$

Variabelsubstitution

Formeln för variabelomvandling.

Kedjeregeln ger $\frac{d}{dx}(G(y(x)) = G'(y(x))\cdot y'(x) = g(y)\frac{dy}{dx}$, integrering map x ger: $G(y(x)) = \int g(y)\frac{dy}{dx}\,dx = \int g\,dy$

Ex

$$\int 2x\cdot\sin x^2\,dx = \begin{bmatrix} t = x^2 & \Rightarrow x = \sqrt{t} \\ \frac{dt}{dx} = 2x & \Rightarrow dt = 2x\,dx \end{bmatrix} = \int \sin t\,dt = -\cos t + C = -\cos x^2 + C$$

$$\int_0^1 x\sqrt{1-x^2}\,dx = \begin{bmatrix} 1-x^2 = t, & x=0 \Leftrightarrow t=1,\ x=1 \Leftrightarrow t=0 \\ \frac{dt}{dx} = -2x & \Rightarrow dt = -2x\,dx \end{bmatrix} = \frac{-1}{2}\int_1^0 \sqrt{t}\,dt = \frac{1}{2}\int_0^1 t^{\frac{1}{2}}\,dt = \frac{1}{2}\,\frac{t^{\frac{3}{2}}}{3/2}\Big|_0^1 = \frac{1}{3}$$

$= -2(x\,dx)$, $x\,dx = \frac{dt}{-2}$

$$\int \sin^2 x\cdot\cos x\,dx = \begin{bmatrix} \sin x = t \\ \frac{dt}{dx} = \cos x \Rightarrow dt = \cos x\,dx \end{bmatrix} = \int t^2\,dt = \frac{t^3}{3} + C = \frac{\sin^3 x}{3} + C$$

$$\int \sin^2 x\cdot\cos^3 x\,dx = \int \sin^2 x\cdot\cos^2 x\cdot\cos x\,dx = \int \sin^2 x(1-\sin^2 x)\cos x\,dx = \begin{bmatrix} \sin x = t \\ \frac{dt}{dx} = \cos x \Rightarrow dt = \cos x\,dx \end{bmatrix} = \int t^2(1-t^2)\,dt = \int t^2 - t^4\,dt = \frac{t^3}{3} - \frac{t^5}{5} + C = \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C$$

$$\int_0^{1/2} \sqrt{1-x^2}\,dx = \begin{bmatrix} x = \sin t, & x=0 \Leftrightarrow \sin t = 0 \\ \frac{dx}{dt} = \cos t & x=\frac{1}{2} \Leftrightarrow \sin t = \frac{\pi}{6} \end{bmatrix} = \int_0^{\frac{\pi}{6}} \sqrt{1-\sin^2 t}\,\frac{dx}{dt}\,dt = \int_0^{\frac{\pi}{6}} |\cos t|\cdot\cos t\,dt = \int_0^{\frac{\pi}{6}} \cos^2 t\,dt = \int_0^{\frac{\pi}{6}} \frac{1+\cos 2t}{2}\,dt = \frac{t}{2} + \frac{\sin 2t}{4}\Big|_0^{\frac{\pi}{6}} = \frac{\pi}{12} + \frac{\sqrt{3}}{2}\cdot\frac{1}{4} = \frac{\pi}{12} + \frac{\sqrt{3}}{8}$$

Tips

$x = a\cdot\sin t$, $\sqrt{a^2 - a^2\sin^2 t} = a\sqrt{1-\sin^2 t}$

$\sqrt{a^2 - x^2}$

$a\sqrt{1-\left(\frac{x}{a}\right)^2}$, $\frac{x}{a} = \sin t$, $\sin^2 t + \cos^2 t = 1$ $\quad \sqrt{a^2+x^2}$, $x = a\cdot\tan t$, ...

$\frac{\sin^2 t}{\cos^2 t} + 1 = \frac{1}{\cos^2 t}$

$\tan t + 1 = \frac{1}{\cos^2 t}$

Ex

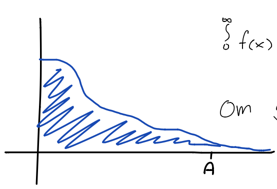
$$\int \frac{e^x + 2}{e^x - e^{-x}}\,dx = \begin{bmatrix} e^x = t \\ \frac{dt}{dx} = e^x \Rightarrow dx = \frac{dt}{e^x} = \frac{dt}{t} \end{bmatrix} = \int \frac{t+2}{t - \frac{1}{t}}\,\frac{dt}{t} = \int\frac{t+2}{t^2-1}\,dt = \int\frac{t+2}{(t-1)(t+1)}\,dt = \int \frac{A}{t+1} + \frac{B}{t-1}\,dt = \int\frac{A(t-1)+B(t+1)}{(t-1)(t+1)}\,dt = \int\frac{At - A + Bt + B}{(t-1)(t+1)}\,dt$$

$$\left.\begin{aligned} t^1&: A + B = 1 \\ t^0&: -A + B = 2 \end{aligned}\right\}\ 2B = 3, \quad A = \frac{3}{2}, \quad B = -\frac{1}{2}$$

$$\int -\frac{1}{2}\cdot\frac{1}{t+1} + \frac{3}{2}\cdot\frac{1}{t-1}\,dt = -\frac{1}{2}\ln|t+1| + \frac{3}{2}\ln|t-1| + C = -\frac{1}{2}\ln|e^x+1| + \frac{3}{2}\ln|e^x-1| + C$$

Arean av begränsade områden Generaliserade (improper) integraler

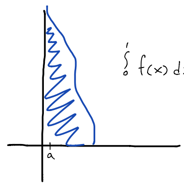
Typ 1:



$$\int_0^\infty f(x)\,dx = \lim_{A\to\infty}\int_0^A f(x)\,dx$$

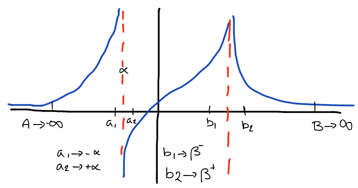
Om gränsvärdet existerar kallas integralen **konvergent**, om inte **divergent**.

Typ 2:



$$\int_0^1 f(x)\,dx = \lim_{a \to 0^+} \int_a^1 f(x)\,dx$$

Blandad:



Ex

$\int_1^\infty \frac{1}{x^p}\,dx$ är konvergent $\Leftrightarrow p>1$

$\int_0^1 \frac{1}{x^p}\,dx$ är konvergent $\Leftrightarrow p<1$

Bevis

$$\int_1^A \frac{1}{x^p}\,dx = \int_1^A x^{-p}\,dx = \frac{x^{1-p}}{1-p}\Big|_1^A = \frac{1}{(1-p)A^{p-1}} - \frac{1}{1-p} \to \infty \text{ om } p<1,\ \to \frac{1}{p-1} \text{ om } p>1$$

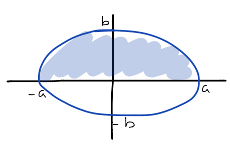
om inte $p=1$: $\int_1^A \frac{1}{x}\,dx = \ln|x|\Big|_1^A = \ln A \to \infty$ om $A \to \infty$

$$\int_a^1 \frac{1}{x^p}\,dx = \frac{x^{1-p}}{1-p}\Big|_a^1 = \frac{1}{1-p} - \frac{a^{1-p}}{1-p} \to \begin{cases} \frac{1}{1-p}, & p<1 \\ \infty, & p>1 \end{cases}$$

$p=1$: $\int_a^1 \ln x\,dx = -\ln a \to \infty,\ a \to 0$

Arean av en ellips

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

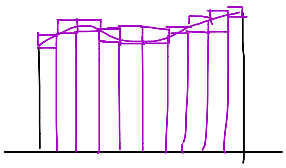


$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$$

$$y = \pm b\sqrt{1-\frac{x^2}{a^2}}$$

$$\text{Arean} = 2\int_{-a}^{a} b\sqrt{1-\frac{x^2}{a^2}}\,dx = 2b\int_{-a}^{a} \frac{1}{a}\sqrt{a^2-x^2}\,dx = \frac{2b}{a}\underbrace{\int_{-a}^{a}\sqrt{a^2-x^2}\,dx}_{\text{arean av en halvcirkel}} = \frac{2b}{a}\cdot\frac{1}{2}\cdot\pi a^2 = \pi ab$$

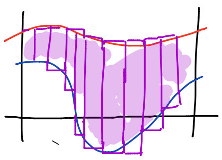
Hur man beräknar.



Numeriska metoder

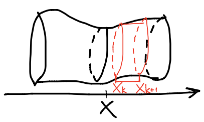
1. Rektangelregeln (mittpunktsregeln) $\int_a^b f(x)\,dx = \sum f\left(\frac{x_k+x_{k-1}}{2}\right)(x_k - x_{k-1})$
2. Trapetsregeln $\int \frac{f(x_k)+f(x_{k-1})}{2}(x_k-x_{k-1})$
3. Simpsons formel (Den MATLAB kör)

Area



Arean $\approx \sum (f(x_k)-g(x_k))(x_k-x_{k-1}) \approx \int_a^b f(x)-g(x)\,dx$

Volym (Skivformeln)

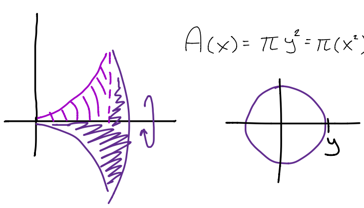


Tvärsnittsarea $A(x)$.

Volymen $\approx \sum A(x_k)(x_{k+1}-x_k)$, detta är Riemannsumman till $\int_a^b A(x)\,dx$

Ex

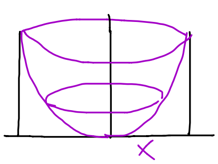
Ett område i planet begränsas av $y=x^2$, $y=0$ och $x=1$. Detta roteras kring x-axeln. Bestäm volymen.



$A(x) = \pi y^2 = \pi(x^2)^2 = \pi x^4$ Volymen blir: $\int_0^1 \pi x^4\,dx = \pi \frac{x^5}{5}\Big|_0^1 = \frac{\pi}{5}$

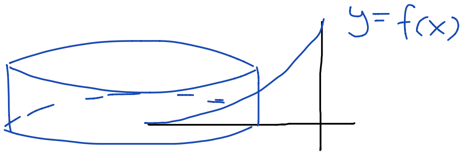
Ex

Samma område kring y-axeln.

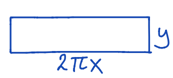


$A(y) = \pi 1^2 - \pi x^2 = \pi - \pi y$ Volymen $= \int_0^1 A(y)\,dy = \int_0^1 \pi - \pi y\,dy = \pi y - \frac{\pi y^2}{2}\Big|_0^1 = \pi - \frac{\pi}{2} = \frac{\pi}{2}$

Rörformeln



Rotationskroppen tänks uppbyggd av tunna cylindriska skal.



Volymen $\approx \sum 2\pi x_k y_k (x_{k+1}-x_k)$, Riemannsumma till: $\int 2\pi x y\,dx$

Rotation kring Y-axeln

$\int_0^1 2\pi x y \, dx = \int_0^1 2\pi x x^2 \, dx = \int_0^1 2\pi x^3 \, dx = \frac{2\pi x^4}{4} \Big|_0^1 = \frac{\pi}{2}$

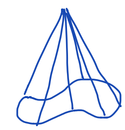
x-axeln

$\int_0^1 2\pi y(1-x)\, dy = \int_0^1 2\pi y(1-y^{\frac{1}{2}})\, dy = \int_0^1 2\pi y - 2\pi y^{\frac{3}{2}} dy = 2\pi\left(\frac{y^2}{2} - \frac{2y^{\frac{5}{2}}}{5}\right)\Big|_0^1 = 2\pi\left(\frac{1}{2} - \frac{2}{5}\right) = \frac{\pi}{5}$

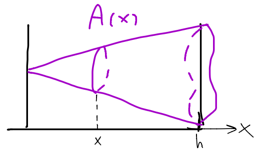
Volymen av en kon

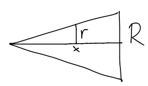
Def

En kon har en spets och en generatris, och består av alla linjer genom spetsen och en punkt i generatrisen.



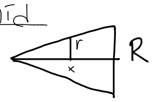
Ex



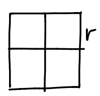


$\frac{r}{x} = \frac{R}{h} \Rightarrow r = \frac{Rx}{h} \Rightarrow A(x) = \pi \frac{R^2x^2}{h^2}$

Pyramid

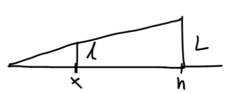


$r = \frac{Rx}{h}$



$A(x) = 2r^2 = 4r^2 = \frac{4R^2x^2}{h^2}$

Allmän kon



$\frac{l}{x} = \frac{L}{h} \Rightarrow l = \frac{Lx}{h}$

Karakteriska längder

KL står i förhållande $\frac{l}{L} = \frac{x}{h}$ areorna i $\left(\frac{l}{L}\right)^2 = \frac{x^2}{h^2}$

Om vi kallar bottenareorna $A(h)$ så har vi $A(x) = \frac{x^2}{h^2} A(h)$

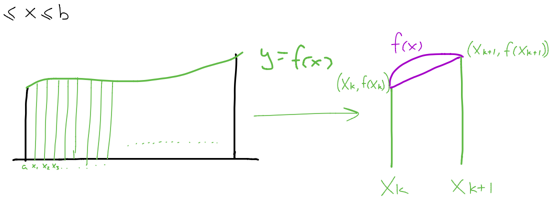
$\int_0^h A(x)\, dx = \int_0^h \frac{x^2}{h^2} A(h)\, dx = \frac{x^3}{3h^2} A(h) \Big|_0^h = \frac{h^3}{3h^2} A(h) = \frac{A(h)h}{3}$

Volymen av ett klot

Ett klot kan fås genom att rotera $y = \sqrt{R^2 - x^2}$ genom x-axeln. Skivformeln ger: $V = \int_{-R}^{R} \pi y^2 dx = \pi \int_{-R}^{R} (R^2 - x^2)\, dx = \pi\left(R^2 x - \frac{x^3}{3}\right)\Big|_{-R}^{R} =$
$\pi\left(R^3 - \frac{R^3}{3} + R^3 - \frac{R^3}{3}\right) = \frac{4\pi}{3} R^3$

Längden av en kurva

$y = f(x), \quad a \leq x \leq b$



Längden av kurvan kan approximeras med polygon.

$\sum_{k=0}^{n-1} \sqrt{(X_{k+1} - X_k)^2 + (f(X_{k+1}) - f(X_k))^2} = \left[\begin{matrix}\text{medelv.}\\ \text{satsen}\end{matrix}\right] = \sum_{k=0}^{n-1} \sqrt{(X_{k+1} - X_k)^2 + (f'(\xi_k))^2 (X_{k+1} - X_k)^2} = \sum_{k=0}^{n-1} \sqrt{(X_{k+1} - X_k)^2 (1 + f'(\xi_k)^2)} =$

$\sum_{k=0}^{n-1} (X_{k+1} - X_k) \sqrt{1 + f'(\xi_k)^2}$, Riemannsumma till $\int_a^b \sqrt{1 + f'(x)^2}\, dx$

Ex

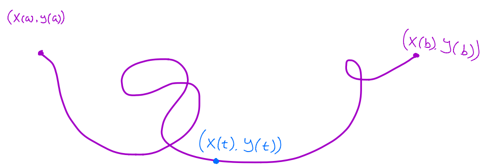
$y = \ln x - \frac{x^2}{8} \qquad 4 \leq x \leq 8$

$$\frac{dy}{dx} = \frac{1}{x} - \frac{2x}{8} = \frac{1}{x} - \frac{x}{4}$$

$$\int_4^8 \sqrt{1 + \left(\frac{1}{x} - \frac{x}{4}\right)^2}\, dx = \int_4^8 \sqrt{1 + \frac{1}{x^2} - \frac{2}{4} + \frac{x^2}{16}}\, dx = \int_4^8 \sqrt{\frac{1}{x^2} + \frac{1}{2} + \frac{x^2}{16}}\, dx = \int_4^8 \sqrt{\left(\frac{1}{x} + \frac{x}{4}\right)^2}\, dx = \int_4^8 \frac{1}{x} + \frac{x}{4}\, dx = \ln x + \frac{x^2}{8} \Big|_4^8 = \ln 8 + 8 - \ln 4 - 4 = \ln 2 + 6$$

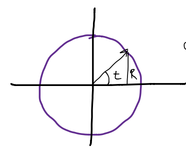
Kurvor, allmänt

Parametriserade kurvor.



Ex

Cirkeln



$$\begin{cases} x = R \cdot \cos(t) \\ y = R \cdot \sin(t) \end{cases}, \quad 0 \leq t \leq 2\pi$$

Gör en indelning av kurvan: $a = t_0 < t_1 < t_2 \ldots < t_n < b$

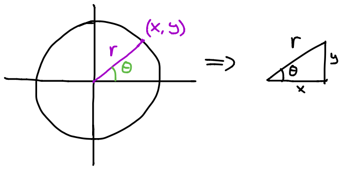
Kurvans längd $\approx \sum_{k=0}^{n-1} \sqrt{(x(t_{k+1}) - x(t_k))^2 + (y(t_{k+1}) - y(t_k))^2} = \ldots\ldots \approx$

$$\int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$$

Ex

Längden av en cirkelbåge, cirkeln $x^2+y^2=r^2$

Parameteriseras:



Längden av en båge:

$x' = -r\sin\theta$

$y' = r\cos\theta$

$\ell = \int_{\theta_1}^{\theta_2} \sqrt{(-r\sin\theta)^2 + (r\cos\theta)^2}\, d\theta$

$= \int_{\theta_1}^{\theta_2} \sqrt{r^2\sin^2\theta + r^2\cos^2\theta}\, d\theta$

$= \int_{\theta_1}^{\theta_2} \sqrt{r^2}\, d\theta$

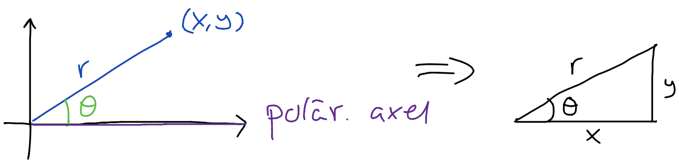
$= r\int_{\theta_1}^{\theta_2} d\theta$

$= r(\theta_2 - \theta_1)$

Hela cirkeln: $0 \leq \theta \leq 2\pi$

$r(2\pi - 0) = 2\pi r$

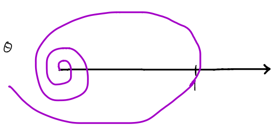
Polära koordinater



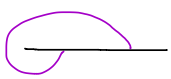
$y = r\sin\theta$

$x = r\cos\theta$

Logaritmisk spiral: $r = e^{-\theta}$



Längden av bågen: $0 \leq \theta \leq 2\pi$



$x = r\cos\theta = e^{-\theta}\cdot\cos\theta$

$y = r\sin\theta = e^{-\theta}\cdot\sin\theta$

$x' = -e^{-\theta}\cdot\cos\theta - e^{-\theta}\cdot\sin\theta$

$y' = -e^{-\theta}\cdot\sin\theta + e^{-\theta}\cdot\cos\theta$

$\ell = \int_0^{2\pi} \sqrt{(-e^{-\theta}\cos\theta - e^{-\theta}\sin\theta)^2 + (-e^{-\theta}\sin\theta + e^{-\theta}\cos\theta)^2}\, d\theta$

$= \int_0^{2\pi} \sqrt{e^{-2\theta}\cos^2\theta + \cancel{2e^{-2\theta}\cos\theta\cdot\sin\theta} + e^{-2\theta}\sin^2\theta - \cancel{2e^{-2\theta}\sin\theta\cos\theta} + e^{-2\theta}\cos^2\theta}\, d\theta$

$= \int_0^{2\pi} \sqrt{e^{-2\theta}(\cos^2\theta + \sin^2\theta + \cos^2\theta + \sin^2\theta)}\, d\theta$

$= \int_0^{2\pi} e^{-\theta}\sqrt{2}\, d\theta$

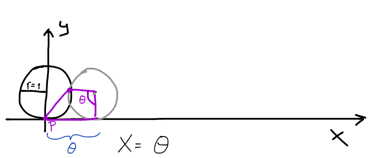
$= -\sqrt{2}\cdot e^{-\theta} \Big|_0^{2\pi}$

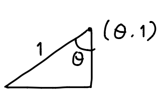
$= \sqrt{2}(-e^{-2\pi} + e^0)$

$= \sqrt{2}\left(1 - \frac{1}{e^{2\pi}}\right)$

Cykloiden

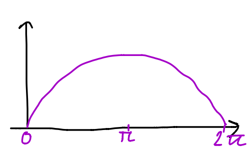
Den kurva en punkt på en cirkel beskriver om cirkeln rullar på en linje.





$\begin{cases} x = \theta - \sin\theta \\ y = 1 - \cos\theta \end{cases}$

Längden av bågen $0 \leq \theta \leq 2\pi$:



$x' = 1 - \cos\theta$

$y' = \sin\theta$

$\ell = \int_0^{2\pi} \sqrt{(1-\cos\theta)^2 + (\sin\theta)^2}\, d\theta$

$= \int_0^{2\pi} \sqrt{1 - 2\cos\theta + \cos^2\theta + \sin^2\theta}\, d\theta$

$= \int_0^{2\pi} \sqrt{2 - 2\cos\theta}\, d\theta$

$= \int_0^{2\pi} \sqrt{4\,\frac{1-\cos\theta}{2}}\, d\theta$

$= \int_0^{2\pi} \sqrt{4\sin^2\frac{\theta}{2}}\, d\theta$

$= \int_0^{2\pi} 2\left|\sin\frac{\theta}{2}\right| d\theta$

$= \int_0^{2\pi} 2\sin\frac{\theta}{2}\, d\theta$

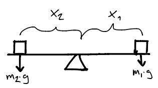
$= 2 \cdot -\cos\left(\frac{\theta}{2}\right)\cdot\frac{1}{\frac{1}{2}} \Big|_0^{2\pi}$

$= 4(-\cos\pi - (-\cos(0)))$

$= 4(1+1) = 8$

Tyngdpunkten av en rak stång

1. Gungbräda

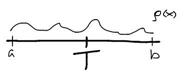


jämvikt: $x_1 \cdot m_1 = x_2 \cdot m_2$

Fler tyngder:

jämvikt: $\sum x_k m_k = \sum y_k n_k$

Kontinuerlig fördelning:



$\rho(x)$ en massfördelning (kg/m)

massan nära x påverkar vridmomentet med $(x-T)\rho(x)\,dx$

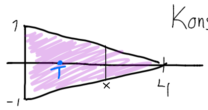
Villkoret för jämvikt: Lika moment vänster som höger kring tyngdpunkten T.

$$\underbrace{\int_a^T (T-x)\rho(x)\,dx}_{\text{Till V.}} = \underbrace{\int_T^b (x-T)\rho(x)\,dx}_{\text{Till H.}}$$

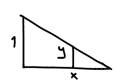
$$\begin{aligned}
0 &= -\int_a^T (T-x)\rho(x)\,dx + \int_T^b (x-T)\rho(x)\,dx \\
&= \int_a^T (x-T)\rho(x)\,dx + \int_T^b (x-T)\rho(x)\,dx \\
&= \int_a^b (x-T)\rho(x)\,dx \\
&= \int_a^b x\cdot\rho(x) - T\rho(x)\,dx \\
&= \int_a^b x\cdot\rho(x)\,dx - \int_a^b T\rho(x)\,dx \\
&= \int_a^b x\cdot\rho(x)\,dx - T\int_a^b \rho(x)\,dx \implies T = \frac{\int_a^b x\cdot\rho(x)\,dx}{\underbrace{\int_a^b \rho(x)\,dx}_{\text{Totala massan}}}
\end{aligned}$$

Ex

Tyngdpunkten för en triangel.



Konstant densitet $\rho \frac{kg}{m^2}$.

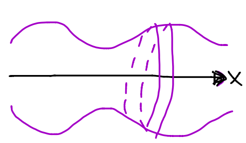


$\frac{y}{1} = \frac{4-x}{4} = 1 - \frac{x}{4}$ $\qquad \rho(x) = 2\left(1-\frac{x}{4}\right)\rho$

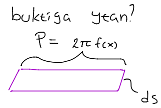
Tyngdpunkten är (T,0) där $T = \frac{\int_0^4 x\cdot 2\left(1-\frac{x}{4}\right)\rho\,dx}{\rho\cdot \text{Arean}} = \frac{1}{2\cdot\frac{4}{2}}\int_0^4 2x - \frac{x^2}{2}\,dx = \frac{1}{4}\left(x^2 - \frac{x^3}{6}\right)\Big|_0^4 = \frac{1}{4}\left(16 - \frac{64}{6}\right) = \frac{16}{4\cdot 3} = \frac{4}{3}$

Rotationsyta

$y = f(x)$ roteras kring x-axeln, vilken area har den buktiga ytan?



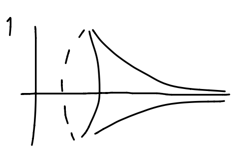
Tänks bestå av band:



$$\text{Arean} = \int_a^b 2\pi f(x)\,ds = \int_a^b 2\pi f(x)\sqrt{1+(f'(x))^2}\,dx$$

Ex

Kurvan $y = \frac{1}{x}$, $x \geq 1$ roteras kring x-axeln



Volymen (Skivformeln) $= \int_1^\infty \pi \frac{1}{x^2}\,dx = -\pi\frac{1}{x}\Big|_1^\infty = 0 + \frac{\pi}{1} = \pi$

Arean: $\int_1^\infty 2\pi\cdot\frac{1}{x}\sqrt{1+\left(-\frac{1}{x^2}\right)^2}\,dx = \int_1^\infty 2\pi\cdot\frac{1}{x}\sqrt{1+\frac{1}{x^4}}\,dx > \int_1^\infty \frac{2\pi}{x}\,dx = \infty$

5.3 13

Riemannsumma: $\sum_{k=0}^{n-1} f(c_k)(x_{k+1}-x_k)$ $x_k \leqslant c_k \leqslant x_{k+1}$

$c_k = \frac{\pi k}{n} = (x_{k+1})$

$x_{k+1} - x_k = \frac{\pi}{n}$

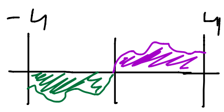
$\sum_{i=1}^{n} \frac{\pi}{n} \cdot \sin\frac{\pi i}{n} \longrightarrow \int \sin(x)\,dx$

$i = 1,\quad c_1 = \frac{\pi}{n} \to 0$ om $n \to \infty$

$i = n,\quad c_n = \frac{\pi n}{n} = \pi$

5.4 13

$\int_{-4}^{4} e^x - e^{-x}\,dx,\quad f(x) = e^x - e^{-x}$ är udda. $f(-x) = e^{-x} - e^{x} = -(e^x - e^{-x}) = f(-x)$



Areorna tar ut varandra.

5.5 41

$\frac{d}{dx}\int_{x^2}^{0} \frac{\sin(t)}{t}\,dt$

Låt $F(x)$ vara s.a. $F'(x) = \frac{\sin(x)}{x}$, t.ex. $\int_0^x \frac{\sin(t)}{t}\,dt$

$\frac{d}{dx}\int_{x^2}^{0} \frac{\sin(t)}{t}\,dt = \frac{d}{dx}\left(F(0) - F(x^2)\right) = 0 - F'(x^2)\cdot 2x = \frac{-\sin(x^2)}{x^2}\cdot 2x = \frac{-2\sin(x^2)}{x}$

49

$\int_{-1}^{1} \frac{1}{x^2}\,dx = \frac{-1}{x}\Big|_{-1}^{1} = \frac{-1}{1} - \left(\frac{-1}{-1}\right) = -1 - 1 = -2$, uppenbarligen fel ty $\frac{1}{x^2} > 0$. Felet är att inter len är generaliserad (typ 2)

$\int_{-1}^{1} \frac{1}{x^2}\,dx = \lim_{a\to 0^-} \int_{-1}^{a} \frac{1}{x^2}\,dx + \lim_{b\to 0^+} \int_{b}^{1} \frac{1}{x^2}\,dx = \lim_{a\to 0^-} -\frac{1}{x}\Big|_{-1}^{a} + \lim_{b\to 0^+} -\frac{1}{x}\Big|_{b}^{1} = \lim_{a\to 0^-} -\frac{1}{a} - 1 - 1 - \lim_{0^+} -\frac{1}{} = +\infty - 1 - 1 + \infty = \infty - 2$

5.6 19

$\int \tan x \cdot \ln(\cos(x))\,dx$

$\int \tan(x)\cdot \ln(\cos(x))\,dx = \int \frac{\sin(x)}{\cos(x)} \cdot \ln(\cos x)\,dx = \begin{bmatrix} \cos x = u \\ \frac{du}{dx} = -\sin x \Rightarrow du\ -\sin x\,dx \end{bmatrix} = -\int \frac{1}{u}\cdot \ln(u)\,du = \begin{bmatrix} \ln(u) = t \\ \frac{dt}{du} = \frac{1}{u} \Rightarrow dt = \frac{du}{u} \end{bmatrix} = -\int t\,dt$

$= \frac{-t^2}{2} = \frac{-\ln^2(\cos x)}{2}$

23

$\int \sin^3 x \cdot \cos^5 x\,dx = \sin^2 x \cdot \sin x \cdot \cos^5 x\,dx = \int \sin x\,(1 - \cos^2 x)\cdot \cos^5 x\,dx = \begin{bmatrix} \cos x = u \\ \frac{du}{dx} = -\sin x \Rightarrow du = -\sin x\,dx \end{bmatrix} = -\int (1 - u^2)\cdot u^5\,du = -\int u^5 - u^7\,du =$

$-\frac{u^6}{6} + \frac{u^8}{8} = \frac{-\cos^6 x}{6} + \frac{\cos^8 x}{8}$

eller:

$\int \sin^3 x \cdot \cos^4 x \cdot \cos x\,dx = \begin{bmatrix} \sin x = u \\ \frac{du}{dx} = \cos x \end{bmatrix} = \int u^3 (1 - u^2)^2\,du \ldots$

6.1 35

Pre ex: $\int \sin x \cdot \cos x\,dx = \int \frac{\sin 2x}{2}\,dx = \frac{-\cos 2x}{4}$

eller: $\begin{bmatrix} \cos x = u \\ \frac{du}{dx} = -\sin x \Rightarrow du = -\sin x\,dx \end{bmatrix} = -\int u\,du = -\frac{u^2}{2} = -\frac{\cos^2 x}{2}$

eller: $\begin{bmatrix} \sin x = u \\ \frac{du}{dx} = \cos x \Rightarrow du = \cos x\,dx \end{bmatrix} = \int u\,du = \frac{u^2}{2} = \frac{\sin^2 x}{2}$

$\frac{\sin^2 x}{2} + \frac{\cos^2 x}{2} = \frac{1}{2}$ (varför C spelar roll)

eller, Part int: $I = \sin x \cdot \sin x - \int \cos x \cdot \sin x\,dx = \sin^2 x - I$, $I = \sin^2 x - I$, $2I = \sin^2 x$, $I = \frac{\sin^2 x}{2}$

35

$I_n = \int \frac{1}{(x^2+a^2)^n} dx$

Rationella funktioner => Partialbråk.

$\frac{1}{x-a} \Rightarrow \ln|x-a|$

$\frac{1}{(x-a)^n} \Rightarrow \frac{1}{(x-a)^{n-1}} \cdot \frac{1}{1-n}$

$\frac{x}{a^2+x^2} \Rightarrow \frac{1}{2}\ln(a^2+x^2)$

$\frac{x}{(a^2+x^2)^n} \Rightarrow \frac{1}{2}\cdot\frac{1}{n-1}\ln(a^2+x^2)^{n-1}$

$\frac{1}{a^2+x^2} \Rightarrow \arctan\left(\frac{x}{a}\right)\cdot\frac{1}{a}$

Specialfall a=1. Det allmänna fallet klaras sen med variabelsub: $\frac{x}{a}=t$

$$I_{n-1} = \int \frac{1}{(x^2+1)^{n-1}} dx = \begin{bmatrix}\text{Part.}\\ \text{int.}\end{bmatrix} = \frac{x}{(x^2+1)^{n-1}} - \int \frac{x(1-n)\cdot 2x}{(x^2+1)^n} dx$$
$$= \frac{x}{(x^2+1)^{n-1}} + 2(n-1)\int \frac{x^2+1-1}{(x^2+1)^n} dx = \frac{x}{(x^2+1)^{n-1}} + 2(n-1)\int \frac{1}{(x^2+1)^{n-1}} dx - \int \frac{1}{(x^2+1)^n} dx$$
$$I_{n-1} = \frac{x}{(x^2+1)^{n-1}} + 2(n-1)\cdot I_{n-1} - 2(n-1)\cdot I_n$$
$$2(n-1) = \frac{x}{(x^2+1)^{n-1}} + (2n-3)I_{n-1}$$
$$I_n = \frac{x}{(2n-2)(x^2+1)^{n-1}} + \frac{2n-3}{2n-2}\cdot I_{n-1}$$

Bestäm I_3

$I_1 = \int \frac{1}{1+x^2} dx = \arctan(x)$

$I_2 = \frac{x}{2(x^2+1)} + \frac{1}{2}\arctan(x)$

$I_3 = \frac{x}{4(x^2+1)^2} + \frac{3}{4}\left(\frac{x}{2(x^2+1)} + \frac{1}{2}\arctan x\right) = \frac{x}{4(x^2+1)^2} + \frac{3x}{8(x^2+1)} + \frac{3}{8}\arctan x$

6.3 22

$\int \frac{1}{(4x-x^2)^{3/2}} dx$

$$\int_{1/2}^{3/2} \frac{1}{(4x-x^2)^{3/2}} dx = \int_{1/2}^{3/2} \frac{1}{(-(x-2)^2+4)^{3/2}} dx = \begin{bmatrix} x-2=u & , \frac{1}{2}=-\frac{3}{2}, \ \frac{3}{2}=-\frac{1}{2} \\ \frac{du}{dx}=1 & \Rightarrow dx=du \end{bmatrix} = \int_{-3/2}^{-1/2} \frac{1}{(4-u^2)^{3/2}} du = \begin{bmatrix} u=2\sin t \\ \frac{du}{dt}=2\cos t \end{bmatrix} = \int_{\arcsin(-\frac{3}{4})}^{\arcsin(-\frac{1}{4})} \frac{2\cos t}{(\sqrt{4-4\sin^2 t})^{3/2}} dt = \int_{arc(-\frac{3}{4})}^{arc(-\frac{1}{4})} \frac{2\cos t}{(2\cos t)^3} dt = \int_{arc(-\frac{3}{4})}^{arc(-\frac{1}{4})} \frac{1}{4\cos^2 t} dt = \frac{1}{4}\Big(\tan t\Big]_{arc\,-\frac{3}{4}}^{arc\,-\frac{1}{4}}$$

$$= \frac{1}{4}\tan(\arcsin(-\tfrac{1}{4})) - \frac{1}{4}\cdot\tan(\arcsin(-\tfrac{3}{4}))$$

31

$$\int \frac{1+x^{1/2}}{1+x^{1/3}} dx = \begin{bmatrix} t=x^{1/6} \\ \frac{dt}{dx}=\frac{1}{6}x^{-5/6} \Rightarrow dx = dt\cdot 6x^{5/6}\end{bmatrix} = \int \frac{1+t^3}{1+t^2}\cdot 6t^5 dt = 6\int \frac{t^8+t^5}{t^2+1} dt$$

$$\begin{array}{l} \quad t^6-t^4+t^3+t^2-t-1 \\ t^8+t^5 \ \big|\ t^2+1 \\ -(t^8+t^6) \\ \quad -t^6+t^5-t^2 \\ \quad \vdots \\ \quad t+1 \end{array}$$

$$= 6\int t^6-t^4+t^3+t^2-t-1+\frac{t+1}{t^2+1} dt$$
$$= 6\left(\frac{t^7}{7}-\frac{t^5}{5}+\frac{t^4}{4}+\frac{t^3}{3}-\frac{t^2}{2}-t+\frac{1}{2}\ln(t^2+1)+\arctan(t)\right., \quad t = x^{1/6}$$

$$\int \frac{t}{t^2+1} dt + \int \frac{1}{t^2+1} dt = \frac{1}{2}\ln(t^2+1) + \arctan t$$

6.5 46

$\Gamma(x) = \int_0^\infty t^{x-1}\cdot e^{-t} dt$

a) Konvergent för $x>0$, okej i ∞. e^{-t} avtar snabbare än alla potenser. I 0 är Γ generaliserad om $x-1<0$, $x<1$ men det är okej om $x-1<1$, $x>0$.

b) $\Gamma(x+1) = \int_0^\infty t^x\cdot e^{-t} dt = \begin{bmatrix}\text{Part}\\ \text{int}\end{bmatrix} = -t^x e^{-t}\Big]_0^\infty + \int_0^\infty x\cdot t^{x-1}\cdot e^{-t} dt = 0 + x\,\Gamma(x)$

c) $\Gamma(1) = \int_0^\infty t^0\cdot e^{-t} dt = -e^{-t}\Big]_0^\infty = -0+1 = 1$

$\Gamma(2) = 1\cdot\Gamma(1) = 1$

$\Gamma(3) = 2\cdot\Gamma(2) = 2$

$\Gamma(4) = 3\cdot\Gamma(3) = 6$

$\Gamma(n) = (n-1)!$

Ordinära, linjära, diffekv. av första ordningen

$a(x)y' + b(x)y = g(x)$. $y(x)$ söks.

Linjär: Om y_1 resp y_2 löser DE med högerled g_1 resp g_2 så löser cy_1 (c konstant) DE med H.L. cg_1 och $y_1 + y_2$ DE med H.L. $g_1 + g_2$.

Bevis

$a(y_1+y_2)' + b(y_1+y_2) = a(y_1'+y_2') + by_1 + by_2 = ay_1' + by_1 + ay_2' + by_2 = g_1 + g_2$

$a(cy_1)' + b(cy_1) = cay_1' + cby_1 = c(ay_1' + by_1) = cg_1$

Lösningsmetod

Ex: $xy' + y = e^x$

$xy' + y = e^x$

$(xy)' = e^x$

$xy = \int e^x dx = e^x + C$

$y = \frac{e^x + C}{x}$

Teori

$y' + f(x)y = g(x)$

Finn F så $F' = f$. Multiplicera med den integrerande faktorn e^F.

$e^{F(x)} \cdot y' + e^{F(x)} \cdot f(x) y = e^{F(x)} \cdot g(x)$

$(e^{F(x)} \cdot y)' = e^{F(x)} \cdot g(x)$

$e^{F(x)} y = \int e^{F(x)} \cdot g(x)\, dx$

$y = e^{-F(x)} \int e^{F(x)} \cdot g(x)\, dx$

Ex

$y' + \frac{1}{x} \cdot y = x$, $y(1) = 1$

Integrerande faktor: $\int \frac{1}{x} dx = \ln x$

$e^{\ln(x)} = x$

$y' + \frac{1}{x} \cdot y = x$

$xy' + y = x^2$

$(xy)' = x^2$

$xy = \frac{x^3}{3} + C$

$1 \cdot 1 = \frac{1}{3} + C \Leftrightarrow C = \frac{2}{3}$

$xy = \frac{x^3}{3} + \frac{2}{3}$

$y = \frac{x^2}{3} + \frac{2}{3x}$

Ex 2

$xy' + 2x^2 y = xe^{-x^2}$

$xy' + 2x^2 y = xe^{-x^2}$, dela med $x \neq 0$.

$y' + 2xy = e^{-x^2}$

$IF = e \wedge \int 2x\, dx = e^{x^2}$

$e^{x^2} \cdot y' + e^{x^2} 2xy = e^{x^2} \cdot e^{-x^2} = 1$

$(e^{x^2} \cdot y)' = 1$

$e^{x^2} \cdot y = \int 1\, dx = x + C$

$y = \frac{x + C}{e^{x^2}} = e^{-x^2} \cdot x + e^{-x^2} \cdot C$

Separabla ekvationer

Antag att ekv är av typ: $f(y) \cdot \frac{dy}{dx} = g(x)$ Allt som innehåller y på ena sidan. x på andra.

Kedjeregeln: $F(y) = G(x) + C$

Detta kan skrivas: $\int f(y) dy = \int g(x) dx$ vilket antyder att det bekväma betraktningssättet: $f(y) \frac{dy}{dx} = g(x)$

$f(y)\, dy = g(x)\, dx$

$\int f(y)\, dy = \int g(x)\, dx$

Ex $\frac{dy}{dx} = -\frac{x}{y}$

$$\frac{dy}{dx} = -\frac{x}{y}$$
$$y\,dy = -x\,dx$$
$$\int y\,dy = \int x\,dx$$
$$\frac{y^2}{2} = -\frac{x^2}{2} + C$$
$$\frac{y^2}{2} + \frac{x^2}{2} = C$$
$$y^2 + x^2 = 2C = D$$
$$y = \pm\sqrt{D - x^2}$$

Enkla tillväxtmodeller

I. Exponentiell tillväxt. (Malthusisk tillväxt)

Antag tillväxthastigheten proportionerlig mot populationen. $x'(t) = cx(t)$, $x(0) = x_0 \Rightarrow \quad X(t) = X_0 \cdot e^{ct}$

II. Logistisk tillväxt.

$$x'(t) = cx(t) \underbrace{- Dx(t^2)}_{\text{Konkurrensterm}}$$

$$x'(t) = D\left(\frac{c}{D}x - x^2\right) = Dx(t)\left(\overset{M}{\frac{c}{D}} - x(t)\right) = Dx(t)(M - x(t))$$

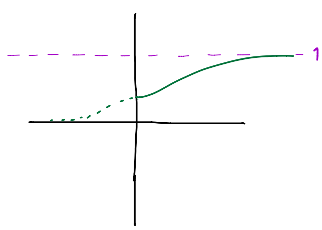
Detta kan fungera som modell för smittspridning eller informationsspridning. $X(t) =$ de smittade
$M - x(t) =$ de icke smittade

Ex $x' = x(1-x)$

$$x' = x(1-x)$$
$$\frac{dx}{dt} = x(1-x)$$
$$\frac{dx}{x(1-x)} = dt$$
$$\int \frac{1}{x(1-x)}\,dx = t + c$$
$$\int \frac{A}{x} + \frac{B}{1-x}\,dx = t + c$$
$$\int \frac{A(1-x) + Bx}{x(1-x)}\,dx = t + c \qquad \left[\begin{array}{l} A = 1 \\ -A + B = 0 \Rightarrow B = 1 \end{array}\right]$$
$$\int \frac{1}{x} + \frac{1}{1-x}\,dx = t + c$$

inre derivata

$$\ln|x| - \ln|1-x| = t + c$$
$$\ln x - \ln(1-x) = t + c \qquad \left[\begin{array}{l} \text{Om } 0 < x(0) < 1 \\ \text{fortsätter } x(t) \in (0,1) \end{array}\right]$$
$$\ln \frac{x}{1-x} = t + c$$
$$\frac{x}{1-x} = e^{t+c} = e^t \cdot e^c = e^t \cdot D$$
$$X = D \cdot e^t(1-x)$$
$$X = De^t - De^t \cdot x$$
$$X + XDe^t = De^t$$
$$X(1 + De^t) = De^t$$
$$X = \frac{De^t}{1 + De^t}$$



$$\lim_{t \to \infty} X(t) = 1 \quad , \quad x(t) < 1$$

III. Linjära ODE av 2:a ordningen

$$a(x)y'' + b(x)y' + c(x)y = g(x)$$

(Linjär: Se bevis för första ordningens)

Skriv ekvationen som $L(y) = g$. Då gäller att:

SATS

Om y_1 är en lösning så är även $y_1 + y$ en lösning om $L(y) = 0$. (y löser den homogena ekvationen.)
Det omvända gäller också: Om y_1 och y_2 löser $y_1 - y_2$ den homogena ekvationen.

Bevis

$L(y_1+y)=L(y_1)+L(y)=g+0=g$

$L(y_1-y_2)=L(y_1)-L(y_2)=g-g=0$

Betrakta nu ekvationer med konstanta koefficienter. $a(x)=a$, $b(x)=b$, $c(x)=c$. Vi söker lösningar till den homogena ekv. $ay''+by'+cy=0$

gissa: $y=e^{rx}$
$y'=r\cdot e^{rx}$
$y''=r^2 e^{rx}$

$ar^2e^{rx}+bre^{rx}+ce^{rx}=0$

$e^{rx}(ar^2+br+c)=0$

r måste lösa den karakteristiska ekvationen: $ar^2+br+c=0$. Om r_1 & r_2 är lösn till KE så blir $Ae^{r_1x}+Be^{r_2x}$ lösn till homogena ekv.

Ex $y''+2y'-3y=1$, $y(0)=y'(0)=0$

$y''+2y'-3y=1$

Homogena ekvationen: $y''+2y'-3y=0$

Karakteristiska ekvationen: $r^2+2r-3=0 \Rightarrow r=-1\pm\sqrt{1+3}=-1\pm2=\begin{matrix} r_1=1 \\ r_2=-3\end{matrix} \Rightarrow y=Ae^{x}+Be^{-3x}$

Hitta nu en partikulär lösning: gissa: $y=c$
$y'=0$, $y''=0$
$-3c=1$
$c=-\frac{1}{3}$

Alla lösningar: $y=-\frac{1}{3}+Ae^{x}+Be^{-3x}$

Begynnelsevärden: $y(0)=-\frac{1}{3}+A+B=0$
$y'(0)=A-3B=0$
$\begin{cases} A=3B \\ 4B=\frac{1}{3} \end{cases} \Rightarrow B=\frac{1}{12} \Rightarrow A=\frac{1}{4} \Rightarrow y(x)=-\frac{1}{3}+\frac{1}{4}e^{x}+\frac{1}{12}e^{-3x}$

Högerled	Partikulär lösning
Konstant	(annan) konstant
Ae^{bx}	ae^{bx}
Ae^{bx}	axe^{bx} om e^{bx} löser den homogena ekv.
Polynom av grad n	Polynom av grad n (samma grad)
$A\cos bx + B\sin bx$	$a\cos bx + c\sin bx$
$e^{ax}(A\cos bx + B\sin bx)$	$e^{ax}(C\cos bx + D\sin bx)$

Ex

$y''+2y'-3y=e^{3x}$

$y''+2y'-3y=e^{3x}$

Vi har redan löst den homogena ekvationen: $y=Ae^{x}+Be^{-3x}$

Partikulär lösning: $y=a\cdot e^{3x}$

$y'=3ae^{3x}$

$y''=9a\cdot e^{3x}$

$y''+2y'+3y=9a\cdot e^{3x}+6ae^{3x}-3a\cdot e^{3x}=12a\cdot e^{3x}=e^{3x}$

$12a=1 \iff a=\frac{1}{12}$

Allmann lösning: $\frac{1}{12}\cdot e^{3x}+Ae^{x}+Be^{-3x}$

Ex

$y''+2y'-3y=x^2$

Partikulär lösning: $y=ax^2+bx+c$

$y'=2ax+b$

$y''=2a$

$y''+2y'-3y=2a+2(2ax+b)-3(ax^2+bx+c)=2a+2b-3c+x(4a-3b)-3ax^2=x^2$

$-3a=1$

$4a-3b=0$ $\qquad a=-\frac{1}{3}$, $3b=4a=-\frac{4}{3} \Rightarrow b=-\frac{4}{9}$, $3c=2a+2b=-\frac{2}{3}-\frac{8}{9}=-\frac{14}{9} \Rightarrow c=-\frac{14}{27}$

$2a+2b-3c=0$

Allmän lösning: $y=-\frac{1}{3}x^2-\frac{4}{9}x-\frac{14}{27}+Ae^{x}+Be^{-3x}$

Ex

$y''+2y'-3y=\sin(2x)$

Partikulärlösning: $y=a\cdot\cos 2x+b\cdot\sin(2x)$

$y'=-2a\sin 2x+2b\cos 2x$

$y''=-4a\cos 2x-4b\sin 2x$

$y''+2y'-3y=-4a\cos 2x-4b\sin 2x+2(\cdots)=\cos 2x(-4a+4b-3a)+\sin 2x(-4b-4a-3b)=\sin 2x$

Allmän: $y=\frac{1}{65}(-4\cos 2x-7\sin 2x)+Ae^{x}+Be^{-3x}$

Ex

$y''+2y'-3y=e^{3x}+\sin 2x$

Partikulär I: $y=\frac{1}{12}e^{3x}$

Partikulär II: $y=\frac{1}{65}(-4\cos 2x-7\sin 2x)$

Totalt: I + II

Krångel med homogena lösningar

Dubbelrot

Karakteristiska ekvationen: $(r-a^2)=0=r^2-2a+a^2$, e^{ax} är fortfarande en giltig lösning men vi behöver en till för att få alla oberoende lösningar. $\Rightarrow y=x\cdot e^{ax}$

$y''=-2ay'+a^2y=0$

$y'=e^{ax}+ax\cdot e^{ax}$

$y''=ae^{ax}+ae^{ax}+a^2xe^{ax}$

$y''-2ay'+a^2y=2ae^{ax}+a^2xe^{ax}-2a(e^{ax}+axe^{ax})+a^2xe^{ax}=2ae^{ax}+a^2xe^{ax}-2ae^{ax}-2a^2xe^{ax}+a^2xe^{ax}=0$

Hom lösning: $Ae^{ax}+Bxe^{ax}=(A+Bx)e^{ax}$

Komplexa rötter

DEF:

$e^{a+ib} = e^{a} \cdot e^{ib} = e^{a}(\cos b + i \sin b)$

$\frac{d}{dt} e^{(a+ib)t} = \frac{d}{dt} e^{at}(\cos bt + i \sin bt) = a e^{at}(\cos bt + i\sin bt) + e^{at}(-b \sin bt + ib\cos bt) = -e^{at}(\cos bt(a+ib) + \sin bt \underbrace{(ai-b)}_{i(a+ib)}) =$

$e^{at}(\cos bt + i \sin bt)(a+ib) = e^{t(a+ib)}(a+ib)$

Alltså

Om KE har komplexa rötter r_1 och r_2 så har den homogena ekvationen lösn: $y = C_1 \cdot e^{r_1 x} + C_2 \cdot e^{r_2 x}$ med komplexa C_1 & C_2.

Om KE (och därmed diffekv) har reella koeff. så är $r_2 = \overline{r_1}$. $r_{1,2} = a \pm ib$

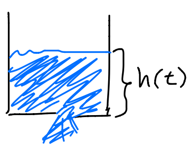
$y = C_1 e^{(a+ib)x} + C_2 e^{(a-ib)x} = e^{ax}(C_1 e^{ibx} + C_2 e^{-ibx})$ om vi startade med en reell ekv. och alltså söker reella lösningar så låter vi imaginärdelarna ta ut varandra. DVS: $C_2 e^{-ibx} = \overline{C_1 \cdot e^{ibx}} = \overline{C_1} \cdot \overline{e^{ibx}} \Leftrightarrow C_2 = \overline{C_1}$

$C_1 = A + iB$, $C_2 = A - iB$

$y = e^{ax}((A+iB)e^{ibx} + (A-iB)e^{-ibx}) = e^{ax}((A+iB)(\cos bx + i\sin bx) + (A-iB)(\cos bx - i \sin bx)) =$

$e^{ax}(A\cos bx - B\sin bx + i(B\cos bx + A\sin bx) + A\cos bx - B\sin bx + i(-B\cos bx - A\sin bx)) = e^{ax}(2A\cos bx - 2B\sin bx)$

Ex Vatten rinner ur burk.



Torricellis lag

Förlorar potentiell energi: mgh

Avgående kinetisk energi: $\frac{mv^2}{2}$

v = utströmningshastighet

$mgh = \frac{mv^2}{2} \Leftrightarrow v = \sqrt{2gh}$

Utströmning: V = volym = $B \cdot h$

$V' = k \cdot A \cdot v$

(A: hålets area)

$-B \cdot h' = k \cdot A \sqrt{2gh}$

$h' = -\frac{kA\sqrt{2g}}{B} \cdot \sqrt{h}$ $\quad \frac{kA\sqrt{2g}}{B} = C$

$\frac{dh}{dt} = -C \cdot h^{\frac{1}{2}}$

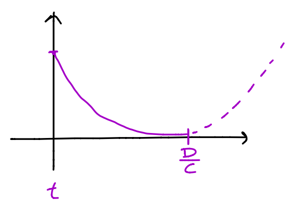
$\frac{dh}{h^{1/2}} = -C\,dt$

$2\sqrt{h} = -Ct + D$

$h = \left(\frac{-Ct+D}{2}\right)^2$

$h(0) = h_0$

$h_0 = \frac{D^2}{4}$



När $h \to 0$ blir modellen dålig, men det finns en annan lösning till DE. | lösningen till v får vi inte deka med $h^{\frac{1}{2}}$ om $h=0$.

Sätt $h(t) = 0$ för att skarva ihop olika lösningar.

Burken tom när $h = 0 \Rightarrow -Ct + D = 0$

$t = \frac{D}{C} = \frac{2\sqrt{h_0}}{C}$

Den fysikaliskt relevanta lösningen: $h(t) = \begin{cases} \left(\frac{D-Ct}{2}\right)^2, & 0 \leq t \leq \frac{D}{C} \\ 0, & t > \frac{D}{C} \end{cases}$

Allmänt om ODE

$y' = F(x,y)$

den här ekv. bestämmer ett riktningsfält.

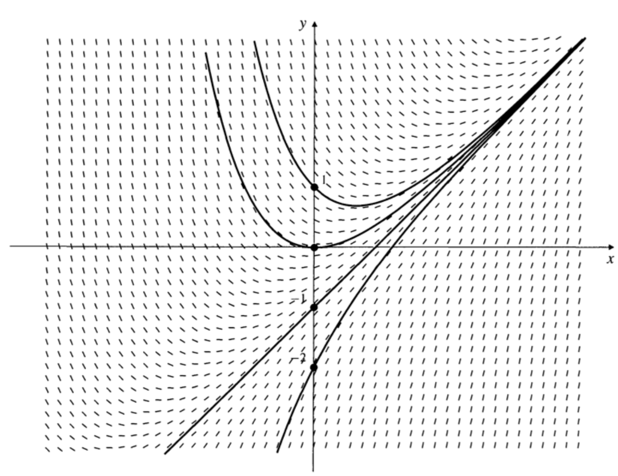
$F(x,y)$ ger en tangentriktning i varje punkt (x,y)

Ex

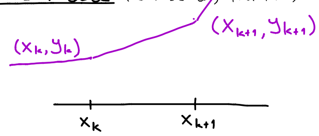
$y' = x - y$ (Fig 17.1 s. 947)

En lösning: $y = x - 1$

$y' = 1 = x - (x-1)$



Eulers metod (en-stegs, framåt)



$\frac{y_{k+1} - y_k}{x_{k+1} - x_k} = F(x_k, y_k)$, $y_{k+1} = y_k + (x_{k+1} - x_k) \cdot F(x_k, y_k)$

7.9-20

$y' + \cos(x) \cdot y = 2xe^{-\sin(x)}$, $y(\pi) = 0$

IF: $\int \cos(x)\,dx = \sin(x) + C$

$e^{\sin x} \cdot y' + e^{\sin x} \cdot \cos x \cdot y = 2xe^{\sin x - \sin x} = 2x$

$(e^{\sin x} \cdot y)' = 2x$

$e^{\sin x} \cdot y = \int 2x\,dx$

$e^{\sin x} \cdot y = x^2 + C$

$y = \frac{x^2 + C}{e^{\sin(x)}}$

$y(\pi) = \frac{\pi^2 + C}{1} = 0 \Rightarrow C = -\pi^2$

$y(x) = \frac{x^2 - \pi^2}{e^{\sin(x)}}$

7.9-22

$y(x) = 1 + \int_0^x \frac{(y(t))^2}{1+t^2}\,dt$

$y'(x) = 0 + \frac{y(x)^2}{1+x^2}$

$\frac{dy}{dx} = \frac{y^2}{1+x^2}$

$\frac{1}{y^2}\,dy = \frac{1}{1+x^2}\,dx$

$-\frac{1}{y} = \arctan(x) + C$

$-1 = y(\arctan(x) + C)$

$y = \frac{-1}{\arctan(x) + C}$

$y(0) = 1 + \int_0^0 \frac{y(0)^2}{1+0^2}\,dt = 1$

$\frac{-1}{\arctan(0) + C} = \frac{-1}{C} = 1$

$C = -1$

$y(x) = \frac{-1}{\arctan(x) - 1}$

17.5-15

$x^2 y'' - xy' + 2y = 0$, $x > 0$

Substituera $t = \ln x$

$x^2 \frac{d^2y}{dx^2} - x\frac{dy}{dx} + 2y = 0$

$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{dy}{dt} \cdot \frac{1}{x}$

$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{dy}{dt} \cdot \frac{1}{x}\right) = \left(\frac{d}{dx} \cdot \frac{dy}{dt}\right)\frac{1}{x} - \frac{dy}{dt} \cdot \frac{1}{x^2} = \frac{d^2y}{dt^2} \cdot \frac{1}{x} \cdot \frac{1}{x} - \frac{dy}{dt} \cdot \frac{1}{x^2}$

$x^2 \cdot \frac{d^2y}{dt^2} - x\frac{dy}{dx} + 2y = \frac{d^2y}{dt^2} - \frac{dy}{dt} - \frac{dy}{dt} + 2y = 0$

KE: $r^2 - 2r + 2 = 0 \Rightarrow r = 1 \pm \sqrt{1-2} = 1 \pm i$

$y(t) = e^t(A\cos t + B\sin t)$

$y(x) = x(A\cos(\ln x) + B\sin(\ln x))$

17.6-4

$y'' + y' - 2y = e^x$

Homo: $y'' + y' - 2y = 0$

KE : $r^2 + r - 2 = 0$

$r = -\frac{1}{2} \pm \sqrt{\frac{1}{4} + \frac{8}{4}} = -\frac{1}{2} \pm \frac{3}{2} \Rightarrow \begin{matrix} r_1 = -2 \\ r_2 = 1 \end{matrix}$

Homogena lösn: $y(x) = Ae^x + Be^{-2x}$

Obs att $y = c \cdot e^x$ inte kan vara en partikulär lösning ty den är del av den homogena lösningen.

Sätt $y = cxe^x$

$y = cxe^x$

$y' = ce^x + cxe^x$

$y'' = ce^x + ce^x + cxe^x$

$y'' + y' - 2y = 2ce^x + cxe^x + ce^x + cxe^x - 2cxe^x = e^x$

$3ce^x + xe^x(c + c - 2c) = e^x$

$3ce^x = e^x$

$c = \frac{e^x}{3e^x} = \frac{1}{3} \Rightarrow y(x) = \frac{1}{3}xe^x + Ae^x + Be^{-2x}$

17.6-9

$y'' + 2y' + 2y = e^{x}\sin(x)$

KE: $r^2 + 2r + 2 = 0$

$r = -1 \pm \sqrt{1-2} = -1 \pm i$

Homo: $y(x) = e^{-x}(A\cos(x) + B\sin(x))$

Partikulärlösning:

Sätt: $y = e^{x}(a\cos x + b\sin x)$

$y' = e^{x}(a\cos x + b\sin x - a\sin x + b\cos x) = e^{x}((a+b)\cos x + (b-a)\sin x)$

$y'' = e^{x}((a+b)\cos x + (b-a)\sin x - (a+b)\sin x + (b-a)\cos x) = e^{x}(2b\cos x - 2a\sin x)$

$y'' + 2y' + 2y = e^{x}(2b\cos x - 2a\sin x + 2(a+b)\cos x + 2(b-a)\sin x + 2a\cos x + 2b\sin x) = e^{x}((4b+4a)\cos x + (4b-4a)\sin x) = e^{x}\sin x$

$\begin{cases} 4b + 4a = 0 \\ 4b - 4a = 1 \end{cases}$ $\quad 8b = 1 \Rightarrow b = \frac{1}{8} \Rightarrow a = -\frac{1}{8}$

Allmän lösning: $y(x) = e^{x}\frac{1}{8}(\sin x - \cos x) + e^{-x}(A\cos x + B\sin x)$

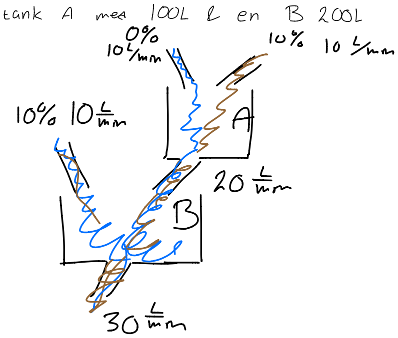
17.6-10

$y'' + 2y' + 2y = e^{-x}\sin x$

Samma VL som innan. Vi vet att vi kommer få resonans $\Rightarrow$ $y = e^{-x}\cdot x(a\cos x + b\sin x)$

Veckoblad

En vattentank A med 100L & en B 200L



Koncentrationerna i A resp B $x(t)$ resp $y(t)$.

Smutsbalans i A: $100x'(t) = 10 \cdot \frac{1}{10} - 20x$

$x'(t) = \frac{1}{100} - \frac{x}{5}$

$x' + \frac{x}{5} = \frac{1}{100}$

IF: $e^{0{,}2x}$

$e^{0{,}2t}\cdot x' + 0{,}2e^{0{,}2t}\cdot x = \frac{e^{0{,}2t}}{100}$

$(e^{0{,}2t}\cdot x)' = \frac{e^{0{,}2t}}{100}$

$e^{0{,}2t}\cdot x = \frac{e^{0{,}2t}}{20} + C$

$x(0) = 0 = \frac{1}{20} + C \Rightarrow C = -\frac{1}{20}$

$x = \frac{1}{20} - \frac{e^{-0{,}2t}}{20}$

Smutsbalans i B: $200y' = 10 \cdot \frac{1}{10} + x \cdot 20 - y \cdot 30$

$200y' + 30y = 1 + 20x = 1 + 1 - e^{-0{,}2t}$

$200y' + 30y = 2 - e^{-0{,}2t}$

$y' + \frac{3}{20}y = \frac{1}{100} - \frac{1}{200}\cdot e^{-0{,}2t}$

IF: $e^{\frac{3}{20}t} = e^{0{,}15t}$

$e^{0{,}15t}\cdot y' + 0{,}15e^{0{,}15t}\cdot y = \frac{e^{0{,}15t}}{100} - \frac{1}{200}\cdot e^{-0{,}05t}$

$(e^{0{,}15t}\cdot y)' = \frac{e^{0{,}15t}}{100} - \frac{1}{200}e^{-0{,}05t}$

$e^{0{,}15t}\cdot y = \frac{e^{0{,}15t}}{15} + \frac{e^{-0{,}05t}}{10} + C$

$y(0) = 0 = \frac{1}{15} + \frac{1}{10} + C \Rightarrow C = \frac{-5}{30} = \frac{-1}{6}$

$y = \frac{1}{15} + \frac{e^{-0{,}2t}}{10} - \frac{e^{-0{,}15t}}{6}$

$\lim_{t\to\infty} x(t) = \frac{1}{20}$ $\quad \left(\frac{1}{2}\cdot 10\%\right)$

$\lim_{t\to\infty} y(t) = \frac{1}{15}$ $\quad \left(\frac{2}{3}\cdot 10\%\right)$

$\frac{2}{3}\cdot\frac{1}{10} = \frac{1}{15}$

Sönderfall

$A \rightarrow B+C$ N atomer av A $N(t) = N(0) \cdot e^{-2kt}$

$B \rightarrow D+E$ y atomer av B, $y(0)=0$ B sönderfaller enl. $y' = -ky$

Vi får alltså att $y' = -ky - N'(t)$, $N'(t) = N(0)e^{-2kt}(-2k)$

$y' = -ky + 2k \cdot N(0)e^{-2kt}$

$y' + ky = 2kN(0)e^{-2kt}$

IF: e^{kt}

$(e^{kt} \cdot y)' = 2kN(0)e^{-2kt}$

$e^{kt} \cdot y = \frac{-2kN(0)e^{-tk}}{k} + C$

$y(0) = 0 \Rightarrow \quad 0 = -2N(0) + C \Rightarrow \quad y = -2N(0)e^{-2kt} + 2N(0)e^{-kt}$

$y = 2N(0)(e^{-kt} - e^{-2kt})$

y har maximum där $y' = 0$

$-k \cdot e^{-kt} + 2k \cdot e^{-2kt} = 0$

$-1 + 2e^{-kt} = 0$

$e^{-kt} = \frac{1}{2}$

$t = \frac{-(\ln\frac{1}{2})}{k} = \frac{\ln 2}{k}$

Hur man gör om en n:te ordningens ODE till 1a ordningens. Inför en vektorbehant.

Ex

$x^2y''-xy'+2y=\sin x$

$y''-\frac{1}{x}y'+\frac{2}{x^2}y=\frac{\sin x}{x^2}$

Inför $u=\begin{bmatrix} y \\ y' \end{bmatrix}$

$u'=\begin{bmatrix} y \\ y' \end{bmatrix}'=\begin{bmatrix} y' \\ y'' \end{bmatrix}=\begin{bmatrix} y' \\ \frac{\sin x}{x^2}-\frac{y'}{x}-\frac{2y}{x^2} \end{bmatrix}=\begin{bmatrix} u_2 \\ \frac{\sin x}{x^2}-\frac{u_2}{x}-\frac{2u_1}{x^2} \end{bmatrix}=F(u,x)$

Komplexa tal

är 2-vektorer som utrustats med en produkt: $\mathbb{R}^2\times\mathbb{R}^2\to\mathbb{R}^2$. Dessutom finns konjugering av komplexa tal.

Om $z=x+iy$, $\bar{z}=x-iy$

Räkneregler

$\overline{z+w}=\bar{z}+\bar{w}$

$\overline{zw}=\bar{z}\,\bar{w}$

$z\bar{z}=(x+iy)(x-iy)=x^2+iyx-iyx+(-1)y^2=x^2+y^2=|z|^2$

$z+\bar{z}=x+iy+x-iy=2x=2\cdot\mathrm{Re}\,z$

$z-\bar{z}=(x+iy)-(x-iy)=x+iy-x+iy=2i\cdot\mathrm{Im}\,z$

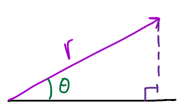
Ex

$\frac{1+3i}{2+3i}=\frac{(2-3i)(1+3i)}{(2-3i)(2+3i)}=\frac{2+6i-3i+9}{2^2+3^2}=\frac{11+3i}{13}=\frac{11}{13}+i\frac{3}{13}$

Skalärprodukt

$\mathrm{Re}\,z\bar{w}=\mathrm{Re}(x+iy)\overline{(u+iv)}=\mathrm{Re}(x+iy)(u-iv)=\mathrm{Re}(xu+iuy-ixv+yv)=xu+yv$

Polära koordinater



$r=|z|$

$x=r\cdot\cos\theta$

$y=r\cdot\sin\theta$

$x+iy=r\cos\theta+ir\sin\theta=r(\cos\theta+i\sin\theta)=re^{i\theta}$

Vi har tidigare visat att $e^{i\theta}e^{i\varphi}=e^{i(\theta+\varphi)}$. Multiplikation $re^{i\theta}Re^{i\varphi}=rRe^{i(\theta+\varphi)}$

$\begin{cases} |zw|=|z||w| \\ \arg(zw)=\arg(z)+\arg(w) \end{cases}$

Ex

$\frac{(1+i\sqrt{3})^{17}}{(1-i\sqrt{3})^{17}}$

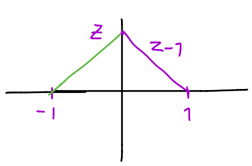
$1+i\sqrt{3}=\left[\begin{matrix}\text{Räkna ut} \\ \text{längden.}\end{matrix}\ (\sqrt{3})^2+1^2=4\Rightarrow|1+\sqrt{3}|=2\right]=2\Big(\underset{\cos\theta}{\underset{\uparrow}{\frac{1}{2}}}+\underset{\sin\theta}{\underset{\uparrow}{\frac{i\sqrt{3}}{2}}}\Big)=2e^{i\frac{\pi}{3}}$

$1-i\sqrt{3}=2\left(\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)=2e^{-i\frac{\pi}{3}}$

$\frac{(1+i\sqrt{3})^{17}}{(1-i\sqrt{3})^{17}}=\frac{2^{17}e^{i\frac{17\pi}{3}}}{2^{17}e^{-i\frac{17\pi}{3}}}=e^{i\frac{34\pi}{3}}=e^{i\frac{4\pi}{3}}=e^{-\frac{2\pi}{3}}$ $\quad\left(2\pi=\frac{6\pi}{3}\right)$

Geometri

Vad är den geometriska betydelsen av $|z-1|=|z+1|$?

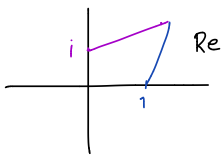


$|z-1|=\text{avst}(z,1)$

$|z+1|=|z-(-1)|=\text{avst}(z,-1)$

Samma avstånd: $\mathrm{Re}\,z=0$

Vad är den geometriska betydelsen av $|z-1| = |z-i|$?



Re z = Im z

Alt. lösn

$z = x+iy$

$|z-1| = |z-i|$

$|z-1|^2 = |z-i|^2$

$|x+iy-1|^2 = |x+iy-i|^2$

$(x-1)^2 + y^2 = x^2 + (y-1)^2$

$x^2 - 2x + 1 + y^2 = x^2 - 2y + 1 + y^2 \Rightarrow -2x = -2y \Leftrightarrow x = y$

Ex

$|z-1| = 2|z+1|$?

$z = (x+iy)$

$|z-1|^2 = 4|z+1|^2$

$|x+iy-1|^2 = 4|x+iy+1|^2$

$(x-1)^2 + y^2 = 4((x+1)^2 + y^2)$

$x^2 - 2x + 1 + y^2 = 4(x^2 + 2x + 1) + 4y^2$

$3x^2 + 10x + 3 + 3y^2 = 0$

$x^2 + \frac{10}{3}x + 1 + y^2 = 0$

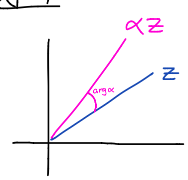
$\underbrace{\left(x + \frac{5}{3}\right)^2}_{x^2 + \frac{10}{3}x + \frac{25}{9}} + y^2 = -1 + \frac{25}{9} = \frac{16}{9} \Rightarrow$ resultatet är en cirkel med centrum i $-\frac{5}{3}$ och radie $\sqrt{\frac{16}{9}} = \frac{4}{3}$.

Multiplikation med α, $|\alpha| = 1$

$|\alpha z| = |\alpha||z| = |z|$

$\arg(\alpha z) = \arg\alpha + \arg z$

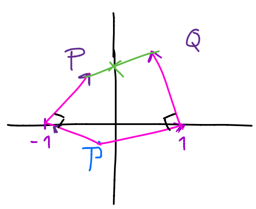
Vridning $\arg(\alpha)$ runt O.



Sjörövarskatten

Infor ett komplext talplan: Palmer i ±1.

Galgen $\mathcal{T}$



$P = -1 + (-i)(-1 - \mathcal{T})$

$Q = 1 + i(1 - \mathcal{T})$

Skatten $= \frac{P+Q}{2} = \frac{1}{2}\left(-1 + i + i\mathcal{T} + 1 + i - i\mathcal{T}\right) = \frac{2i}{2} = i$

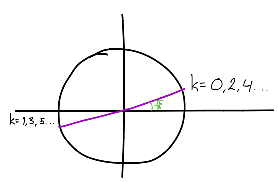
Kvadratrötter

Lös $z^2 = 1+i$

$z = re^{i\theta}$

$z^2 = r^2 e^{2i\theta} = 1 + i = \sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right) = \sqrt{2}\, e^{i\frac{\pi}{4}}$

$\begin{cases} r^2 = \sqrt{2} \\ 2\theta = \frac{\pi}{4} + 2k\pi \end{cases} \Rightarrow \begin{matrix} r = \sqrt[4]{2} \\ \theta = \frac{\pi}{8} + k\pi \end{matrix}$



$z = \pm\sqrt[4]{2}\, e^{i\frac{\pi}{8}}$

Lösn. 2

$z = x+iy$

$z^2 = (x+iy)^2 = x^2 + 2ixy - y^2 = 1+i$

$\left.\begin{matrix} x^2 - y^2 = 1 \\ 2xy = 1 \end{matrix}\right\} \; y = \frac{1}{2x} \Rightarrow x^2 - \frac{1}{4x^2} = 1 \Rightarrow 4x^4 - 4x^2 - \frac{1}{4} = 0$

$[x^2=u]$

$u^2-u-\frac{1}{4}=0$

$u=\frac{1}{2}\pm\sqrt{\frac{1}{4}+\frac{1}{4}}=\frac{1}{2}\pm\sqrt{\frac{1}{2}}=$ $\quad (u=x^2>0)$

$x=\pm\sqrt{\frac{1}{2}+\frac{1}{\sqrt{2}}}$

$y=\pm\frac{1}{2\sqrt{\frac{1}{2}+\frac{1}{\sqrt{2}}}}$

Ex

$z^2=2+3i$

$z=re^{i\theta}$

$z^2=r^2e^{2i\theta}=2+3i=\sqrt{13}\left(\frac{2}{\sqrt{13}}+i\frac{3}{\sqrt{13}}\right)=\sqrt{13}\,e^{i\arctan\frac{3}{2}}$

$$\left.\begin{aligned} r^2&=\sqrt{13} && \Rightarrow\ r=\sqrt[4]{13} \\ 2\theta&=\arctan\tfrac{3}{2}+2k\pi && \Rightarrow\ \theta=\tfrac{1}{2}\arctan\tfrac{3}{2}+k\pi \end{aligned}\right\}\ z=\pm\sqrt[4]{13}\,e^{i\frac{1}{2}\arctan\frac{3}{2}}$$

Högre rötter

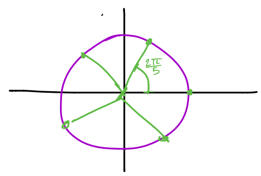
Lös $z^5=1$

$z=re^{i\theta}$

$z^5=r^5e^{5i\theta}=1=1\cdot e^{i\cdot 0}$

$r^5=1 \quad \Rightarrow \quad r=1$

$5\theta=0+2k\pi \quad \Rightarrow \quad \theta=\frac{2}{5}k\pi$



Ex

$z^3=8i$

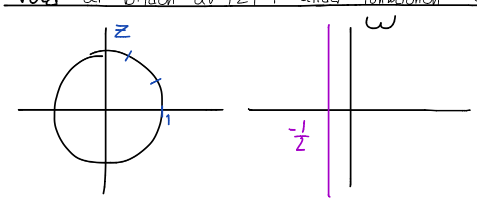
$z=re^{i\theta}$

$z^3=r^3e^{3i\theta}=8i=8e^{i\frac{\pi}{2}}$

$r^3=8 \qquad r=2$

$3\theta=\frac{\pi}{2}+2k\pi \qquad \theta=\frac{\pi}{6}+\frac{2k\pi}{3}$

Vad är bilden av $|z|=1$ under funktionen $w=\frac{1}{z-1}$?



Punkter med $|1|=e^{i\theta}=z$

$w=\frac{1}{e^{i\theta}-1}=\frac{1}{\cos\theta+i\sin\theta-1}=\frac{\cos\theta-1-i\sin\theta}{(\cos\theta-1)^2+\sin^2\theta}=\frac{\cos\theta-1-i\sin\theta}{\cos^2\theta-2\cos\theta+1+\sin^2\theta}=\frac{\cos\theta-1-\sin\theta}{2-2\cos\theta}=$

$\frac{\cos\theta-1}{2(1-\cos\theta)}-i\frac{\sin\theta}{2(1-\cos\theta)}=-\frac{1}{2}-i\frac{\sin\theta}{2(1-\cos\theta)}$

$\frac{\sin\theta}{2(1-\cos\theta)}=\frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cdot 2\cos^2\frac{\theta}{2}}=\frac{1}{2}\tan\frac{\theta}{2} \qquad -\infty<\tan\frac{\theta}{2}<\infty$

Definition av e^{iy}

$e^{iy} = \cos y + i\sin y$

Sant: $e^{iy} \cdot e^{ix} = e^{i(x+y)}$

Def

$e^{z} = e^{x+iy} = e^{x} \cdot e^{iy} = e^{x}(\cos y + i\sin y)$

$e^{-iy} = \cos y - i\sin y$

Ex

$$\int \cos^4 x\,dx = \int \left(\frac{e^{ix}+e^{-ix}}{2}\right)^4 dx = \frac{1}{16}\int e^{4ix} + 4e^{(3-1)ix} + 6e^{(2-2)ix} + 4e^{(1-3)ix} + e^{-4ix}\,dx = \frac{1}{16}\int e^{4ix} + 4e^{2ix} + 6 + 4e^{-2ix} + e^{-4ix}\,dx = \frac{1}{16}\left[\frac{e^{4ix}}{4i} + \frac{4e^{2ix}}{2i} + 6x + \frac{4e^{-2ix}}{-2i} + \frac{4e^{-4ix}}{-4i}\right] =$$

$$\frac{1}{16}\left(\underbrace{\frac{e^{4ix}-e^{-4ix}}{4i}}_{\frac{\sin 4x}{2}} + 6x + 4\underbrace{\frac{e^{2ix}-e^{-2ix}}{2i}}_{\sin(2x)}\right) = \frac{1}{32}\sin(4x) + \frac{3}{8}x + \frac{1}{4}\sin(2x) + C$$

DEF

$\cos z = \frac{e^{iz}+e^{-iz}}{2}$ (Sant om $z \in \mathbb{R}$)

$\sin z = \frac{e^{iz}-e^{-iz}}{2i}$ $\quad$ — $\quad$ " $\quad$ —

Ex

$\cos z = 2$

$\frac{e^{iz}+e^{-iz}}{2} = 2$

$\left[w = e^{iz} \Rightarrow \frac{w+\frac{1}{w}}{2} = 2,\ w + \frac{1}{w} = 4,\ w^2 + 1 = 4w,\ w^2 - 4w + 1 = 0 \qquad w = 2 \pm \sqrt{4-1} = 2 \pm \sqrt{3} \right]$

$e^{iz} = 2 + \sqrt{3}$

$z = x + iy$

$e^{ix-y} = e^{ix} \cdot e^{-y} = (2+\sqrt{3}) =$

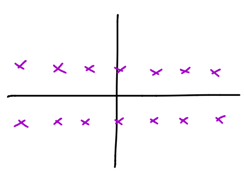
Belopp: $e^{-y} = (2+\sqrt{3})e^{i0}$

Arg : $x = 0 + 2k\pi$

$e^{iz} = 2 - \sqrt{3}$

$e^{-y} = 2 - \sqrt{3}$

$x = 0 + 2k\pi$



$\frac{1}{2-\sqrt{3}} = \frac{2+\sqrt{3}}{(2+\sqrt{3})(2-\sqrt{3})} = \frac{2+\sqrt{3}}{4-3} = 2 + \sqrt{3}$

$y = \begin{cases} -\ln(2-\sqrt{3}) \\ -\ln(2+\sqrt{3}) \end{cases}$

$x = 0 + 2k\pi$

Ex

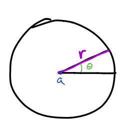
Visa att $w = \frac{1}{z-1}$ avbildar cirkeln $|z| = 2$ på en cirkel med centrum $\frac{1}{3}$.

Bevis

$$\left|w - \frac{1}{3}\right|^2 = \left|\frac{1}{z-1} - \frac{1}{3}\right|^2 = \left|\frac{3-z+1}{3(z-1)}\right|^2 = \left[z = 2e^{i\theta}\right] = \frac{1}{9}\left|\frac{4-2e^{i\theta}}{2e^{i\theta}-1}\right|^2 = \frac{4}{9}\left|\frac{2-e^{i\theta}}{2e^{i\theta}-1}\right|^2 = \frac{4}{9}\left|\frac{2-\cos\theta - i\sin\theta}{2\cos\theta - 1 + 2i\sin\theta}\right|^2 = \frac{4}{9}\left(\frac{(2-\cos\theta)^2 + \sin^2\theta}{(2\cos\theta-1)^2 + 4\sin^2\theta}\right) = \frac{4}{9}\left(\frac{4-4\cos\theta+\cos^2\theta+\sin^2\theta}{4\cos^2\theta - 4\cos\theta + 1 + 4\sin^2\theta}\right) =$$

$$\frac{4}{9}\left(\frac{5-4\cos\theta}{5-4\cos\theta}\right) = \frac{4}{9}.$$

$\left|w - \frac{1}{3}\right| = \frac{2}{3}$, cirkel med centrum i $\frac{1}{3}$ och $r = \frac{2}{3}$



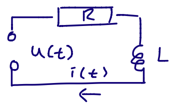
Cirkel med centrum a och radie r.

$z - a = re^{i\theta}$

$z = a + re^{i\theta}$

Komplexa impedanser

$U = Ae^{j\omega t}$, $j^2 = -1$



$u(t) = Ri(t) + j\omega L \cdot i(t)$

konstant fasförskjutning

$$i(t) = \frac{u(t)}{R+j\omega L} = \frac{Ae^{j\omega t}}{R+j\omega L} = \frac{Ae^{j\omega t}}{\sqrt{R^2+\omega^2L^2}\cdot e^{j\arctan(\frac{\omega L}{R})}} = \frac{A}{\sqrt{R^2+\omega^2L^2}} \cdot e^{j(\omega t - \arctan\frac{\omega L}{R})}$$

Frekvensberoende amplitud

samma frekvens

Taylorpolynom

DEF

Taylorpolynomet p av grad n i a till $f(x)$ är det polynom som uppfyller $p^{(i)}(a) = f^{(i)}(a)$ $i = 0,1,2,\ldots,n$

Ex $n=3$

$P(x) = C_3(x-a)^3 + C_2(x-a)^2 + C_1(x-a) + C_0$

$p(a) = C_0 = f(a)$

$P'(x) = 3C_3(x-a)^2 \cdot 1 + 2C_2(x-a) + C_1$

$P'(a) = C_1 = f'(a)$

$P''(x) = 6C_3(x-a) + 2C_2$

$P''(a) = 2C_2 = f''(a)$

$P'''(x) = 6C_3$

$P'''(a) = 6C_3 = f'''(a)$

$$P(x) = \frac{f'''(a)}{3\cdot 2}(x-a)^3 + \frac{f''(a)}{2}(x-a)^2 + f'(a)(x-a) + f(a)$$

Allmänt:

$$P(x) = f(a) + f'(a)(x-a) + f''(a)\frac{(x-a)^2}{2} + \ldots + \frac{f^{(k)}(a)(x-a)^k}{k!} + f^{(n)}(a)\frac{(x-a)^n}{n!}$$

Felterm

integrations-konstant

$$f(x) - f(a) = \int_a^x f'(t)\,dt = \int_a^x 1 \cdot f'(t)\,dt = \Big[(t-x)f'(t)\Big]_a^x - \int_a^x (t-x)f''(t)\,dt = 0 - (a-x)\cdot f'(a) - \left[\frac{(t-x)^2}{2}\cdot f''(t)\right]_a^x + \int_a^x \frac{(t-x)^2}{2} f'''(t)\,dt =$$

$$(x-a)f'(a) - 0 + \frac{(a-x)^2}{2}f''(a) + \left[\frac{(t-x)^3}{3\cdot 2}\cdot f'''(t)\right]_a^x - \int_a^x \frac{(t-x)^3}{3\cdot 2} f'''(t)\,dt = (x-a)f'(a) + \frac{(x-a)^2}{2}f''(a) + 0 - \frac{(a-x)^3}{3!}f'''(a) - \left[\frac{(t-x)^4}{4!}f^{IV}(t)\right]_a^x + \int_a^x \frac{(t-x)^4}{4!}f^{IV}(t)\,dt =$$

$$(x-a)f'(a) + \frac{(x-a)^2}{2}f''(a) + \frac{(x-a)^3}{3!}f'''(a) - 0 + \frac{(a-x)^4}{4!}f^{IV}(a) + \ldots + \frac{(x-a)^k}{k!}f^{(k)}(a) + \underbrace{(-1)^k \int_a^x \frac{(t-x)^k}{k!} f^{(k+1)}(t)\,dt}_{\text{resttermen i integralform}}$$

Vi skall senare visa Lagranges restterm:

$$(-1)^k \int_a^x \frac{(t-x)^k}{k!} f^{(k+1)}(t)\,dt = \frac{(x-a)^k}{(k+1)!} f^{(k+1)}(\xi) \quad , \; a < \xi < x$$

AI 44 & 45.

Visa att $\overline{\left(\frac{z}{w}\right)} = \frac{\overline{z}}{\overline{w}}$.

$z = x+iy$, $w = u+iv$

$$\overline{\left(\frac{z}{w}\right)} = \overline{\left(\frac{x+iy}{u+iv}\right)} = \overline{\frac{(u-iv)(x+iy)}{(u-iv)(u+iv)}} = \overline{\frac{xu+yv+i(uy-xv)}{u^2+v^2}} = \frac{xu+yv-i(uy-xv)}{u^2+v^2}$$

$$\frac{\overline{z}}{\overline{w}} = \frac{x-iy}{u-iv} = \frac{(x-iy)(u+iv)}{(u-iv)(u+iv)} = \frac{xu+yv+i(xv-yu)}{u^2+v^2}$$

57] Låt $x_1, x_2, \ldots x_n$ vara rötterna till $x^n=1$. Visa att $\sum_{k=1}^{n} x_k = 0$

Bevis

ekv är: $x^n - 1 = 0$

$$x^n - 1 = (x-x_1)(x-x_2)(x-x_3)\ldots(x-x_n) = x^n + \underbrace{x^{n-1}(-x_1-x_2\ldots-x_n)}_{\substack{=0 \\ \text{jmfr med VL.}}} + \text{lägre ordn. termer} \Rightarrow x_1+x_2+\ldots x_n = 0$$

Bevis 2

Geometrisk summa. $S = 1+c+c^2\ldots+c^n$

$$\begin{aligned} cS &= c+c^2+c^3\ldots+c^{n+1} \\ S-cS &= 1-c^{n+1} \\ (1-c)S &= 1-c^{n+1} \\ S &= \frac{1-c^{n+1}}{1-c} \quad c \neq 1 \end{aligned}$$

Lös ekv $x^n = 1$, $x = re^{i\theta} \Rightarrow x^n = r^n e^{i\theta n} = 1 e^{i0}$

$$\begin{cases} r^n = 1 & \Rightarrow r = 1 \\ n\theta = 0 + 2k\pi & \Rightarrow \theta = \frac{2k\pi}{n} \end{cases}$$

$x_k = 1 \cdot e^{\frac{2\pi}{n}(k)i}$ $\quad x_k = \left(e^{\frac{2\pi}{n}i}\right)^k = c^k \quad \Rightarrow \sum_{k=1}^{n} x_k = c+c^2+c^3\ldots+c^n = c(1+c+c^2+\ldots+c^{n-1}) = c\,\frac{1-c^n}{1-c} = e^{\frac{2\pi}{n}i} \cdot \frac{1-\overbrace{e^{\frac{2\pi i}{n}}}^{=1}}{1-c} = 0$

$x^n = 1$

AII 22

Hitta alla komplexa nollställen till $\sin(z)=0$.

$$\sin(z) = \frac{e^{iz}-e^{-iz}}{2} = 0$$

$e^{iz} = e^{-iz} = \frac{1}{e^{iz}}$

$e^{2iz} = 1$

$z = x+iy \Rightarrow e^{2iz} = e^{2xi-2y} = e^{2ix} \cdot e^{-2y} = 1 \cdot e^{i0}$

$e^{-2y} = 1$

$2x = 0+2k\pi$

$y = 0$

$x = k\pi$

26]

$$\cos z = \frac{e^{iz}+e^{-iz}}{2} = [z = x+iy] = \frac{e^{ix-y}+e^{-ix+y}}{2} = \frac{e^{-y}(\cos(x)+i\sin(x))+e^{y}(\cos(x)-i\sin(x))}{2} = \frac{\cos(x)}{2}(e^{-y}+e^{y}) + \frac{i\sin(x)}{2}(e^{-y}-e^{y})$$

$\mathrm{Re}(\cos(z)) = \frac{\cos(x)}{2}(e^{-y}+e^{y})$

$\mathrm{Im}(\cos(z)) = \frac{i\sin(x)}{2}(e^{-y}-e^{y})$

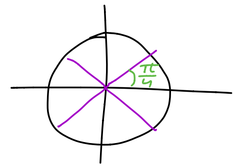
33⌋ $P(z) = z^4 + 1$

Faktorisera $P(z)$ i reella faktorer.

$z^4 = -1$ saknar reella lösningar. Om vi däremot låter $z = re^{i\theta} \Rightarrow z^4 = r^4 e^{i4\theta} = -1 = 1 e^{i\pi}$

$r^4 = 1 \quad \Rightarrow \quad r = 1$

$4\theta = \pi + 2k\pi \quad \Rightarrow \quad \theta = \frac{\pi}{4} + \frac{k\pi}{2}$



$z = \pm \frac{1}{\sqrt{2}} \pm i \frac{1}{\sqrt{2}}$

$P(z) = \left(z - \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right)\left(z - \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)\left(z + \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right)\left(z + \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)$

$= (z - z_1)(z - \bar{z}_1)(z - z_2)(z - \bar{z}_2)$

$= \left(z^2 + z\left(-\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} - \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2\right)\left(z^2 + z\left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}} + \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2\right)$

$= \left(z^2 - \frac{2}{\sqrt{2}} z + 1\right)\left(z^2 + \frac{2}{\sqrt{2}} + 1\right)$

$= \left(z^2 - \sqrt{2}\, z + 1\right)\left(z^2 + \sqrt{2} + 1\right)$

34⌋ $P(z) = z^4 - 4z^3 + 12z^2 - 16z + 16$, $z_1 = 1 - \sqrt{3}i$

Påstående: $P(z_1) = 0$

Om $P(z_1) = 0 \Leftrightarrow P(\bar{z}_1) = 0$ och detta går att dela med $(z - z_1)(z - \bar{z}_1)$ — $z_1 \bar{z}_1 = |z_1|^2$

$(z - z_1)(z - \bar{z}_1) = (z - 1 + \sqrt{3}i)(z - 1 - \sqrt{3}i) = z^2 + z(-1 + \sqrt{3}i - 1 - \sqrt{3}i) + (1^2 + (\sqrt{3})^2) = z^2 - 2z + 4$

$$
\begin{array}{l|l}
\multicolumn{2}{l}{z^2 - 2z + 4} \\
\hline
z^4 - 4z^3 + 12z^2 - 16z + 16 & z^2 - 2z + 4 \\
\hline
\qquad\qquad\qquad 0 &
\end{array}
$$

Divisionen gick jämt ut $\Rightarrow$ z_1 & $\bar{z}_1$ är nollställen. Återstående nollställen ges av att lösa $z^2 - 2z + 4 = 0$ men vi vet redan att det ger $z = 1 \pm \sqrt{3}i$

$P(z) = (z - 1 - \sqrt{3}i)^2 (z - 1 + \sqrt{3}i)^2 = (z^2 - 2z + 4)^2$

Tenta 2012-08-28

UPP 7.

$x \in \mathbb{R}$, $\frac{x}{x-i}$ blir en cirkel med $c = \frac{1}{2}$, $r = \frac{1}{2}$

Bevis

$\left|\frac{x}{x-i} - \frac{1}{2}\right|^2 = \left|\frac{2x - (x-i)}{2(x-i)}\right|^2 = \frac{1}{4}\left|\frac{x+i}{x-i}\right|^2 = \frac{1}{4} \frac{x^2+1}{x^2+1} = \frac{1}{4}$

2011-03-17

Uppg 6.

Finn real och imaginärdel av $\frac{2 - e^{it}}{2 + e^{it}}$.

$\frac{(2 - e^{it})(2 + e^{-it})}{(2 + e^{it})(2 + e^{-it})} = \frac{4 - 2e^{it} + 2e^{-it} - 1}{4 + 2e^{it} + 2e^{-it} + 1} = \left[\frac{e^{it} + e^{-it}}{2} = \cos t ,\ \frac{e^{it} - e^{-it}}{2i} = \sin t\right] = \left[-2e^{it} + 2e^{-it} = \frac{-4e^{it} + 4e^{-it}}{2} = -4 \frac{e^{it} - e^{-it}}{2} = 4i \frac{e^{it} - e^{-it}}{2i}\right] = \frac{3 - 4i\sin t}{5 + 4\cos t} =$

$\frac{3}{5 + 4\cos t} + i\left(\frac{-4\sin t}{5 + 4\cos t}\right)$

Taylor-utveckling

Ex

e^{3x} kring $x=1$. (Potenser av $x-1$)

$$e^{3x} = e^{3(x-1)+3} = e^3 \cdot e^{3(x-1)} = e^3 \cdot \left(1 + \frac{3(x-1)}{1} + \frac{3(x-1)^2}{2} + \frac{3(x-1)^3}{3!} + \frac{3(x-1)^4}{4!} + \dots\right)$$

Utveckling av $\ln(1+x)$ kring $x=0$

$$1+x+x^2+\dots+x^n = \frac{1-x^{n+1}}{1-x}$$

$$n \to \infty \Rightarrow 1+x+x^2+\dots+x^n+\dots = \frac{1}{1-x} \quad \text{om } |x|<1.$$

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = 1-x+x^2-x^3+x^4-x^5+\dots$$

$$\ln(1+x) = \int \frac{1}{1+x}\,dx = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots + C \quad |x|<1$$

$$0 = \ln(1) = 0+0+\dots+C \Rightarrow C=0$$

Olika former för resttermen

Integralform: $r(x) = (-1^n)\int_a^x \frac{(t-x)^n}{n!} f^{(n+1)}(t)\,dt$

Lagrange: $r(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1} \quad a<\xi<x$

Bevis

En generaliserad medelvärdessats.

Om f och g är kontinuerliga samt att g inte ändrar tecken existerar ett ξ s.a. $\int_a^b f(x)g(x)\,dx = f(\xi)\int_a^b g(x)\,dx$

För fallet $g>0$.

Låt $m = \min_b f(x)$, $M = \max f(x)$. $a \le x \le b$, $m \le f(x) \le M$. $m\,g(x) \le f(x) \le M\,g(x)$

$$\int_a^b m\,g(x)\,dx \le \int_a^b f(x)g(x)\,dx \le \int_a^b M\,g(x)\,dx$$

$$m\int_a^b g(x)\,dx \le \int_a^b f(x)g(x)\,dx \le M\int_a^b g(x)\,dx$$

$$m \le \frac{\int_a^b f(x)g(x)\,dx}{\int_a^b g(x)\,dx} \le M$$

Enligt satsen om mellanliggande värden existerar $\exists\xi \Rightarrow f(\xi) = \frac{\int_a^b fg\,dx}{\int_a^b g\,dx}$

Bevis av Lagrange

$$(-1)^n\int_a^x \frac{(t-x)^n}{n!} f(t)\,dt = f^{(n+1)}(\xi)(-1)^n\int_a^x \frac{(t-1)^n}{n!}\,dt = \left[f^{(n+1)}(\xi)(-1)^n\frac{(t-x)^{n+1}}{(n+1)!}\right]_a^x = f^{(n+1)}(\xi)(-1)^{n+1}\frac{(a-x)^{n+1}}{(n+1)!} = f^{(n+1)}(\xi)\frac{(x-a)^{n+1}}{(n+1)!}$$

Def

f är "stort O" av g när $x\to a$. (Detta skrivs: $f(x) = O(g(x))$.) om $\left|\frac{f(x)}{g(x)}\right|$ är begränsad när $x\to a$

Restterm $r(x) = O(x-a)^{n+1}$

$O(x^k)$ betyder termer x^k och högre potenser.

Ex

$$\lim_{x\to0}\frac{1-\cos(x)}{x^2} = \lim_{x\to0}\frac{1-(1-\frac{x^2}{2}+\frac{x^4}{4!}-\frac{x^6}{6!}\dots)}{x^2} = \lim_{x\to0}\frac{1-(1-\frac{x^2}{2}+O(x^4))}{x^2} = \lim_{x\to0}\frac{1-1+\frac{x^2}{2}+O(x^4)}{x^2} = \lim_{x\to0}\frac{1}{2}+O(x^2) = \frac{1}{2}$$

Ex

$$\frac{\sin^2 x}{1-\cos x} = \frac{(x-O(x^3))^2}{1-(1-\frac{x^2}{2}+O(x^4))} = \frac{x^2-2xO(x^3)+O(x^3)^2}{\frac{x^2}{2}+O(x^4)} = \frac{x^2+O(x^4)+O(x^6)}{\frac{x^2}{2}+O(x^4)} = \frac{x^2+O(x^4)}{\frac{x^2}{2}+O(x^4)} = \frac{x^2(1+O(x^2))}{x^2(\frac{1}{2}+O(x^2))} \to \frac{1}{\frac{1}{2}} = 2$$

Ex

$$\frac{\sin(x)\arctan(x)-x^2}{(\cos(x)-1)^2} = \frac{(x-\frac{x^3}{6}+O(x^5))(x-\frac{x^3}{3}+O(x^5))-x^2}{(1-\frac{x^2}{2}+O(x^4)-1)^2} = \frac{\cancel{x^2}-\frac{x^4}{6}+O(x^5)-\frac{x^4}{3}+\frac{x^6}{18}+O(x^8)+O(x^6)+O(x^8)+O(x^{10})\cancel{-x^2}}{\frac{x^4}{4}+O(x^6)+O(x^8)} = \frac{-\frac{1}{2}x^4+O(x^6)}{\frac{x^4}{4}+O(x^6)} = \frac{x^4(-\frac{1}{2}+O(x^2))}{x^4(\frac{1}{4}+O(x^2))} = \frac{-\frac{1}{2}}{\frac{1}{4}} = -2$$

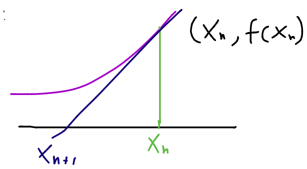
Approximation av e^x

$$r(x) = \frac{e^\xi \cdot x^{n+1}}{(n+1)!} = e^\xi\,\frac{x\cdot x\cdot x\cdot x\dots}{1\cdot2\cdot3\cdot4\cdots(n-1)\cdot n} \to 0 \quad \forall x.$$

Om $f(x) = P(x) + O(x^{n+1})$, p är ett polynom av grad n så är p Maclaurinpolynomet

Feluppskattning i Newtons metod

Metoden:



Tangentens ekv: $y - f(x) = f'(X_n)(X - X_n)$

$y = 0$

$-f(x_n) = f'(x_n)(X_{n+1} - X_n)$

$X_{n+1} = X_n - \frac{f(x_n)}{f'(x_n)}$

Låt X^* vara lösningen till $f(x) = 0$. Lagrange restterm ger då att:

$$0 = f(x^*) = f(x_n) + f'(x_n)(X^* - X_n) + f''(\xi)\frac{(X^* - X_n)^2}{2} =$$

$$-f'(x_n)(X^* - X_{n+1}) = f''(\xi)\frac{(X^* - X_n)^2}{2}$$

$$-f'(X_n)(X_{n+1} - X_n) + f'(X_n)(X^* - X_n) + f''(\xi)\frac{(X^* - X_n)^2}{2} =$$

$$f'(X_n)(-X_{n+1} + X_n + X^* - X_n) + f''(\xi)\frac{(X^* - X_n)^2}{2} = 0$$

$$\Longleftrightarrow$$

$$X^* - X_{n+1} = -\frac{f''(\xi)}{2f'(X_n)}(X^* - X_n)^2$$

$$|X^* - X_{n+1}| \leqslant \frac{\max|f''|}{2\min|f'|}(X^* - X_n)^2$$

En àposterioriuppskattning

Om vi att $f(x_n) \approx f(x^*) = 0$, vad vet vi då om $X_n - X^*$?

$$f(X_n) - f(X^*) = f'(\xi)(X_n - X^*)$$

$$|X_n - X^*| \leqslant \frac{|f(x_n) - f(x^*)|}{\min|f'|}$$

V1

Inverterting av funktioner

Ex

$y = \frac{e^x - e^{-x}}{2}$ (= sinh(x))

$[e^x = u]$

$y = \frac{u - \frac{1}{u}}{2}$

$2y = u - \frac{1}{u}$

$u^2 - 2yu - 1 = 0$

$u = y \pm \sqrt{y^2+1}$

$\sqrt{y^2+1} > y$

$u = y + \sqrt{y^2+1}$ $(u > 0)$

$x = \ln u = \ln(y + \sqrt{y^2+1})$

Ex

$x = \frac{1+y}{1-y}$

$x(1-y) = 1+y$

$x - yx = 1+y$

$x - 1 = y + yx = y(1+x)$

$y = \frac{x-1}{x+1}$

V2

Derivator

Kedje- och produktreglerna.

Ex

$f(x) = (x^2+1)(2x-1)$

$f'(x) = 2x(2x-1) + (x^2+1)2$

Ex

$f(x) = e^{x^2}$

$f'(x) = e^{x^2} \cdot 2x$

Ex

$f(x) = x \cdot e^{x^2}$

$f'(x) = e^{x^2} + x \cdot 2x \cdot e^{x^2} = e^{x^2} + 2x^2 e^{x^2}$

$f''(x) = 2x \cdot e^{x^2} + 4x \cdot e^{x^2} + 2x^2 \cdot 2x \cdot e^{x^2} = 6xe^{x^2} + 4x^3 \cdot e^{x^2}$

Ex

$f(x) = x \cdot e^{x^2 \cdot \sin(x)}$

$f'(x) = e^{x^2 \cdot \sin(x)} + x \cdot e^{x^2 \sin(x)}(2x \cdot \sin(x) + x^2 \cos(x))$

Ex

$f(x) = x^x = e^{\ln(x)^x} = e^{x \cdot \ln(x)}$

$f'(x) = e^{x \cdot \ln(x)}(\ln x + x \cdot \frac{1}{x}) = x^x(\ln(x+1))$

Gränsvärden

$\frac{\sin(x)}{x} \to 1$, $\frac{e^x - 1}{x} \to 1$, $x \to 0$

Dessa tolkas som derivator i 0. $\sin 0 = 0$

$e^0 = 1$

eller så nvänder vi Taylors formel:

$\frac{\sin(x)}{x} = \frac{x+O(x^3)}{x} = 1+O(x^2) \to 1$

$\frac{e^x-1}{x} = \frac{1+x+O(x^2)-1}{x} = 1+O(x) \to 1$

$\lim_{x\to 0} x\cdot\ln(x) = \left[y=\ln(x)\right] = \lim_{y\to -\infty} e^y\cdot y = 0$

Implicit derivering

Ex

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ Sökt: $\frac{dy}{dx}$

Derivering map x.

$\frac{2x}{a^2}+\frac{2y\cdot y'}{b^2}=0$

$\frac{2y\cdot y'}{b^2} = -\frac{2x}{a^2}$

$y' = -\frac{xb^2}{ya^2}$

Primitiva funktioner

Glöm inte:

$\int \frac{1}{x}dx = \ln|x|$

$\int \frac{1}{1+x^2}dx = \arctan x$

$\int \frac{1}{\sqrt{1-x^2}}dx = \arcsin x$

$\int \frac{1}{x+2}dx = \ln|x+2|$

$\int \frac{1}{1-x^2}dx = \int \frac{1}{(1+x)(1-x)}dx = \int \frac{A}{1+x}+\frac{B}{1-x}dx = \int \frac{A(1-x)+B(1+x)}{(1-x)(1+x)}dx = \int \frac{A-Ax+B+Bx}{(1-x)(1+x)}dx = \left[\begin{matrix}A+B=1\\ B-A=0\end{matrix} = A=B=\frac{1}{2}\right] = \frac{1}{2}\int \frac{1}{1+x}+\frac{1}{1-x}dx =$

$\frac{1}{2}\left(\ln|1+x| \underset{\substack{\uparrow\\ \text{inre derivatan}}}{-} \ln|1-x|\right)$

$\int \frac{1}{x^2-x}dx = \int \frac{1}{x(x-1)}dx = \int \frac{A}{x}+\frac{B}{x-1}dx = \int -\frac{1}{x}+\frac{1}{x-1}dx = -\ln|x|+\ln|x-1| = \ln\left|\frac{x-1}{x}\right|$

Kvadratkomplettering

$\int \frac{1}{x^2+2x+2}dx = \int \frac{1}{(x+1)^2+2-1}dx = \int \frac{1}{(x+1)^2+1}dx = \arctan(x+1)$

$\int \frac{1}{2x+3}dx = \frac{1}{2}\ln|2x+3|$

$\int \frac{1}{(2x+3)^2}dx = \int (2x+3)^{-2}dx = \frac{(2x+3)^{-1}}{-1}\cdot\frac{1}{2} = -\frac{1}{2(2x+3)}$

V3

Integraler

Glöm inte inre derivatan vid variabelsubstitution.

Skriv ut dx, du och $\frac{du}{dx}$.

Ex

S: tid mätt i sek

h: tid mätt i tim

$h = \frac{s}{3600}$

$v(s)$ = hastigheten i $\frac{m}{s}$.

Tillryggalagd sträcka: $\int_0^T v(s)\,ds = \left[\begin{matrix}h=\frac{s}{3600}\\ dh=\frac{1}{3600}ds\end{matrix}\right] = \int_0^{\frac{T}{3600}} v(h\cdot 3600)\,3600\,dh$

$\left[\begin{matrix}u=s^2\\ du=2s\,ds\end{matrix}\right] \Rightarrow \int_0^T v(s)\,ds = \int_0^{T^2} v(\sqrt{u})\frac{1}{2s}du = \int_0^{T^2}\frac{v(\sqrt{u})}{2\sqrt{u}}du$

Ex

$I = \int \sin(2x)\cdot\cos(3x)\,dx = \frac{\sin(2x)\cdot\sin(3x)}{3} - \int \cos(2x)\cdot\sin(3x)\cdot\frac{2}{3}dx = \frac{\sin(2x)\sin(3x)}{3}+\frac{2}{9}\cos(2x)\cos(3x)+\frac{2}{9}\int\sin(2x)\cos(3x)\cdot 2\,dx^{*} =$

$\frac{\sin(2x)\sin(3x)}{3}+\frac{2}{9}\cos(2x)\cos(3x)+\frac{4}{9}I$

$\frac{5}{9}I = $ —— 〃 ——

Huvudsatsen

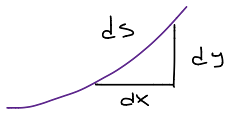
$\int_a^b f(t)\,dt = F(b)-F(a)$

$$\frac{d}{dx}\int_x^{x^2}(y(x))^3\,dx = \frac{d}{dx}\left(F(x^2)-F(x)\right) = \frac{d}{dx}F(x^2)\cdot 2x - \frac{d}{dx}F(x) = \left(y(x^2)\right)^3\cdot 2x - (y(x))^3$$

V4

Tillämpn.

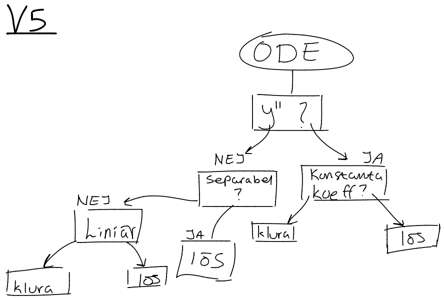
Kom ihåg



$$ds = \sqrt{dx^2+dy^2} = dx\sqrt{1+\left(\frac{dy}{dx}\right)^2} = dt\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}$$

dt, om du har en parametrisering av kurvan

V5



Komplexa tal

Polar form: $z = |z|e^{i\theta} = |z|(\cos\theta + i\sin\theta)$

Ex

$$1-i = \sqrt{1^2+(-1)^2}\left(\frac{1}{\sqrt2} - \frac{i}{\sqrt2}\right) = \sqrt2\, e^{-i\frac{\pi}{4}}$$

Ex

$$-2+3i = \sqrt{13}\left(\frac{-2}{\sqrt{13}} + \frac{3i}{\sqrt{13}}\right) = \sqrt{13}\,e^{i(\arctan(-\frac{3}{2})+\pi)}$$

$$\tan\theta = \frac{\frac{3}{\sqrt{13}}}{\frac{-2}{\sqrt{13}}} = -\frac{3}{2}$$

2013-04-13

1. $\frac{dy}{dx} = \frac{y^2}{x}$ $\quad y(1) = 1$

$\frac{1}{y^2} dy = \frac{1}{x} dx$

$-\frac{1}{y} = \ln x + C$

$-1 = 0 + C \Rightarrow C = -1$

$-\frac{1}{y} = \ln x - 1$

$y = \frac{1}{1 - \ln x}$

2.